

THE IMBEDDING OF THE SKEW PART OF A
BILINEAR FUNCTION IN LINEAR
ASSOCIATIVE ALGEBRAS



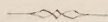
YUAN LAY

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A Dissertation Submitted in Partial Fulfillment
of the Requirements for the Degree of Doctor
of Philosophy, in the University of Michigan.



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§ I. INTRODUCTION

Linear associative algebras have been studied by several authors, such as Th. Molien, J. H. M. Wedderburn, L. E. Dickson, and A. A. Albert.¹ The point of view taken by these authors is that a multiplication is given in a space and they try to split the space into parts (or present it as a direct sum of subspaces) in such a way that the given algebra induces algebras in these subspaces and different algebras are classified in terms of the properties of such subspaces. In this work we shall take a different point of view. For us the algebra is given by a *bilinear function*² and instead of splitting the space into subspaces, we split this bilinear function into parts, the *skew part* and the *symmetric part*,³ and classify the algebras in terms of the properties of these parts. Of course, the object of our study is the same, but it seems that our classification does not coincide with that arrived at from the other point of view except that we also arrive at the *quaternion algebra*, which has been studied in detail, and in fact, was introduced long before the general study of algebras had begun.⁴ Even here, in the quaternion case, our point of view leads us to an expression for quaternion multiplication (which is a special case of our general formula

$$\sigma(x, y) = \lambda_y x + \lambda_{xy} + K_{x, y} + \phi(x, y), \S 5)$$

which is different from that usually given.⁵ The study of the skew part which we shall denote by $\phi(x, y)$ for any two elements x and y of the algebra is formally analogous to a certain

part of the study of infinitesimal transformations of a continuous group (The Theory of Sophus Lie⁶), where $\phi(x,y)$ corresponds to the alternant or Kammersymbol operation (x,y) and the Jacobian Identity $\phi(\phi(x,y),z) + \phi(\phi(y,z),x) + \phi(\phi(z,x),y) = 0$ is the same as

$$((x,y),z) + ((y,z),x) + ((z,x),y) = 0 \quad 7$$

Adopting the definition of linear associative algebras and that of a field from A. A. Albert's Modern Higher Algebra, we shall also call an algebra over a field an *n-dimensional space of vectors* where the *vectors* of the space are the elements of the algebra and the dimension of the space is equivalent to the order of the algebra over the field as defined in the above mentioned book of Albert's, and we call the numbers of the field the *scalars*. Obviously the space is linear, or it is the ordinary linear vector space.⁸ From the same book we take the definitions of commutativity and associativity relative to the multiplication operation, or the bilinear function $\sigma(x,y)$, and also the definition of linear dependence and independence among the elements of the algebra or, in our language, among the vectors of the space. Whenever we say a coordinate system, we mean a set of linearly independent vectors or a basis as defined in the same book of Albert's.

Since we can always write

$$\sigma(x,y) = \frac{\sigma(x,y) + \sigma(y,x)}{2} + \frac{\sigma(x,y) - \sigma(y,x)}{2}$$

where

$$\frac{\sigma(x,y) + \sigma(y,x)}{2}$$

is symmetric and linear in x and y and

$$\frac{\sigma(x,y) - \sigma(y,x)}{2}$$

is skew and linear in x and y , we shall set

$$\theta(x,y) = \frac{\sigma(x,y) - \sigma(y,x)}{2}$$

and

$$\phi(x,y) = \frac{\sigma(x,y) - \sigma(y,x)}{2}$$

Then

$$\sigma(x,y) = \theta(x,y) + \phi(x,y)$$

where

$$\theta(x,y) \text{ and } \phi(x,y)$$

represent respectively the symmetric and the skew part of $\sigma(x,y)$ and both are linear in x and y . Thus

$$\theta(x,y) = \theta(y,x), \quad \phi(x,y) = -\phi(y,x)$$

and in commutative algebras

$$\sigma(x,y) = \sigma(y,x) \text{ or } \phi(x,y) = 0 \text{ for all } x \text{ and } y.$$

In general ϕ -function may not vanish identically, though it always vanishes for two arguments in the same direction. It may vanish in certain number of ways which we can regard as the indication of the degree of non-commutativity of the algebra. Our purpose is to find a σ -operation for a given ϕ -function, such that for any x and y in the space

$$\phi(x,y) = \frac{1}{2}(\sigma(x,y) - \sigma(y,x))$$

and

$$\sigma(\sigma(x,y),z) = \sigma(x,\sigma(y,z))$$

and then to secure a classification of those possible algebras in terms of their different degree of non-commutativity. We shall see that for this purpose the ϕ -function has to satisfy the Jacobian Identity, i. e. the Jacobian Identity is a necessary condition for the existence of the associative σ -operation, but it is not sufficient to give an associative algebra as we shall show in §10.

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1. Th. Molien, Math. Annalen, vol. 41 (1893), P. 113, Ueber System hoherer complexer Zahlen.

J. H. M. Wedderburn }
 L. E. Dickson } For their works and the works of many others
 A. A. Albert } along this line, see the Literaturverzeichnis in
 Max Deuring's Algebren, Ergebnisse der Mathematik
 Und ihrer Grenzgebiete, Vol. 4, No. 1, pp. 138-143

2. By a bilinear function we mean a function of two variables which is linear in both arguments. Here the arguments are vectors of the space or elements of the algebra and the function is a vector function, giving a member of the same space. If we express the vectors in terms of their components corresponding to a coordinate system or a basis as Albert defines in his *Modern Higher Algebra*, the function will be a vector whose components are bilinear forms such as discussed in L. E. Dickson's *Linear Algebras* and M. Bocher's *Introduction to Higher Algebras*.
3. A function is said to be symmetric if an interchange of any two of its arguments does not change the function; a function is said to be skew (or skew-symmetric) if an interchange of any two of its arguments changes the sign of the function. If a skew function is linear in its arguments, then it vanishes when there is a linear dependence among the arguments, due to the fact that a skew function with two equal arguments must be equal to, after an interchange of those two arguments, the same function with sign changed or must be zero. Compare the definitions of symmetric and skew matrices in Albert's *Modern Higher Algebra* and Bocher's *Introduction to Higher Algebras*.
4. W. R. Hamilton, *Trans. Irish Acad.*, vol. 21 (1848), p. 199 (1843). *Lectures on Quaternions*, 1853; *Elements of Quaternions*, 1866, etc.
5. Using the units 1, i, j, k where
- $$i \cdot j = j \cdot k = k \cdot i = 0, \quad i^2 = j^2 = k^2 = -1,$$
- and
- $$i \times j = k, \quad j \times k = i, \quad k \times i = j$$
- We express ordinarily the multiplication of two quaternions
- $$x = x^0 + x^1 i + x^2 j + x^3 k \quad \text{and} \quad y = y^0 + y^1 i + y^2 j + y^3 k$$
- as follows:
- $$x \times y = x \cdot y + x^0 (y^1 i + y^2 j + y^3 k) + y^0 (x^1 i + x^2 j + x^3 k) + x^1 i + x^2 j + x^3 k \wedge (y^1 i + y^2 j + y^3 k)$$
- where:
- $$x \cdot y = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$$
- and $\wedge$ denotes the ordinary vector product in the 3-dimensional vector space; or, what amounts to the same thing, the formulas (247) and (248), p. 246, in Hamilton's paper on quaternions, *Trans. Irish Acad.*, vol. 21 (1848)
6. Sophus Lie, *Theorie der Transformations Gruppen*.
7. J. E. Campbell, *Theory of Continuous Groups*, p. 8, p. 67; N. Jacobson, *A Class of Normal Simple Lie Algebras of Characteristic 0*, *Ann. of Math.*, 2nd series, vol. 38, no. 2, p. 508; Sophus Lie, *Theorie der Transformations Gruppen*, part 1, p. 94
8. For definition of linear vector space and subspaces, see J. V. Neumann's *Functional Operators* (Notes by Dr. Roberts S. Martin, 1934-1935, p. 2-1)

§ 2. PRELIMINARY THEOREMS

Theorem 1. In a linear vector space of dimension n , any two skew linear scalar functions of n vectors differ from each other only by a scalar constant multiplier.

Proof. Take any two sets of n linearly independent vectors

$$x_1, x_2, \dots, x_n \text{ and } y_1, y_2, \dots, y_n$$

Then among the y_i 's, $i=1, 2, \dots, n$,

there must exist one, say y_1 , such that

$$y_1 = \eta_{11}x_1 + \eta_{12}x_2 + \dots + \eta_{1n}x_n$$

where $\eta_{11} \neq 0$, since otherwise the y_i 's would all depend on $n-1$ x_i 's and hence they would not be linearly independent.

Thus,

$$y_1, x_2, x_3, \dots, x_n$$

are linearly independent, and

$$y_1, y_2, \dots, y_n$$

can all be expressed by their linear combinations. Again among $y_2, y_3, \dots, y_n$ there must exist one, say y_2 , such that,

$$y_2 = \eta_{21}y_1 + \eta_{22}x_2 + \eta_{23}x_3 + \dots + \eta_{2n}x_n$$

where $\eta_{22} \neq 0$, since otherwise $y_1, y_2, \dots, y_n$ would all depend on $n-1$ vectors $y_1, x_3, x_4, \dots, x_n$ and hence they would not be linearly independent. Thus

$$y_1, y_2, x_3, x_4, \dots, x_n$$

are linearly independent, and $y_3, y_4, \dots, y_n$ can all be expressed by their linear combinations. Proceeding in this way, we have

$$y_i = \eta_{i1}y_1 + \eta_{i2}y_2 + \dots + \eta_{i,i-1}y_{i-1} + \eta_{ii}x_i + \eta_{i,i+1}x_{i+1} + \dots + \eta_{in}x_n$$

where $\eta_{ii} \neq 0$.

Consider now any two skew linear scalar functions of n vectors, $f(x_1, x_2, \dots, x_n)$ and $g(x_1, x_2, \dots, x_n)$.

We see by substitution

$$\begin{aligned} f(y_1, y_2, \dots, y_n) &= \eta_{nn} f(y_1, y_2, \dots, y_{n-1}, x_n) \\ &= \eta_{n-1, n-1} \eta_{nn} f(y_1, y_2, \dots, y_{n-2}, x_{n-1}, x_n) = \dots \\ &= \eta_{11} \eta_{22} \dots \eta_{nn} f(x_1, x_2, x_3, \dots, x_n) \end{aligned}$$

using the fact that if any two of the arguments of a skew function are collinear the function will vanish. Likewise,

$$g(y_1, y_2, \dots, y_n) = \eta_{11} \eta_{22} \dots \eta_{nn} g(x_1, x_2, \dots, x_n)$$

Therefore

$$\frac{f(y_1, y_2, \dots, y_n)}{f(x_1, x_2, \dots, x_n)} = \eta_{11} \eta_{22} \dots \eta_{nn} = \frac{g(y_1, y_2, \dots, y_n)}{g(x_1, x_2, \dots, x_n)}$$

$$\text{or } f(y_1, y_2, \dots, y_n) = \frac{f(x_1, x_2, \dots, x_n)}{g(x_1, x_2, \dots, x_n)} g(y_1, y_2, \dots, y_n)$$

If we choose a definite set of $x_1, x_2, \dots, x_n$, $\frac{f(x_1, x_2, \dots, x_n)}{g(x_1, x_2, \dots, x_n)}$

will be a constant, say k , hence

$$f(y_1, y_2, \dots, y_n) = k g(y_1, y_2, \dots, y_n)$$

or the theorem.

If a coordinate system is used, the determinant of the components of any n vectors in the n -dimensional space is a skew linear scalar function of the n vectors. Since any other skew linear scalar function of n vectors should differ from this particular one only by a scalar constant multiplier, we shall call the totality of all skew linear scalar functions of n vectors the *determinant functions*. Since the determinant of the components of n linearly independent vectors is not zero, we have

Corollary: A determinant function vanishes if and only

if there is a linear dependence among its arguments.

Theorem 2: If $x_1, x_2, \dots, x_{n+1}$ are $n+1$ vectors in a linear vector space of dimension n , and $[x_1, x_2, \dots, x_n]$ denotes any determinant function of $x_1, x_2, \dots, x_n$, then

$$[x_1, x_2, \dots, x_n] x_{n+1} = [x_{n+1}, x_2, x_3, \dots, x_n] x_1 \\ + [x_1, x_{n+1}, x_3, \dots, x_n] x_2 + \dots + [x_1, x_2, x_3, \dots, x_{n-1}, x_{n+1}] x_n$$

Proof: Suppose $x_1, x_2, \dots, x_n$ are linearly independent. Then we can write

$$x_{n+1} = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n \quad \text{and}$$

$$[x_1, x_2, \dots, x_n] x_{n+1} = [x_1, x_2, \dots, x_n] (\xi_1 x_1 + \dots + \xi_n x_n).$$

Comparing the coefficient of x_1 in the right hand side with the same in the right hand side of the equation to be proven, we see that

$$[x_{n+1}, x_2, x_3, \dots, x_n] x_1 = [\xi_1 x_1, x_2, x_3, \dots, x_n] x_1 \\ + [\xi_2 x_2, x_2, x_3, \dots, x_n] x_1 + \dots + [\xi_n x_n, x_2, \dots, x_n] x_1 \\ = \xi_1 [x_1, x_2, \dots, x_n] x_1$$

where the other terms drop out since their coefficients contain two arguments of same direction and hence vanish by virtue of the skew linear property of the determinant function. Same result comes out for the coefficients of $x_2, x_3, \dots$, and x_n . Hence the equation to be proven for the theorem is shown to be true. Suppose now $x_1, x_2, \dots, x_n$ are not linearly independent. Then

$[x_1, x_2, \dots, x_n] = 0$ as $x_1, x_2, \dots, x_n$ can be expressed by linear combinations of $n-1$ or less linearly independent vectors and hence $[x_1, x_2, \dots, x_n]$ can be expressed by a sum of determinant functions each of which contains two or more arguments of same direction or is zero. If the coefficients of all the terms in the right hand side of the equation to be proven are zeros, then the equation is proved. Suppose the

coefficient of some term is not zero, say the coefficient of x_i , i. e.

$$[x_1, x_2, \dots, x_{i-1}, x_{n+1}, x_{i+1}, \dots, x_n] \neq 0.$$

Then

$$x_1, x_2, \dots, x_{i-1}, x_{n+1}, x_{i+1}, \dots, x_n$$

are linearly independent. The foregoing proof shows that the following equation is true:

$$\begin{aligned} & [x_1, x_2, \dots, x_{i-1}, x_{n+1}, x_{i+1}, \dots, x_n] x_i \\ &= [x_i, x_2, \dots, x_{i-1}, x_{n+1}, x_{i+1}, \dots, x_n] x_1 \\ &+ [x_1, x_i, x_3, \dots, x_{i-1}, x_{n+1}, x_{i+1}, \dots, x_n] x_2 + \dots \\ &+ [x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n] x_{n+1} + \dots \\ &+ [x_1, x_2, \dots, x_{i-1}, x_{n+1}, x_{i+1}, \dots, x_n] x_i \end{aligned}$$

If we transfer all terms in the right hand side except the term $[x_1, x_2, \dots, x_n] x_{n+1}$ to the other side and interchange x_i and x_{n+1} in the determinant functions, with the sign of those terms changing twice, we see that this is the same equation which we are to prove for the theorem. In other words, the equation of the theorem is true. Hence the theorem.

Theorem 3. If a determinant function, not identically zero, vanishes identically for one of its arguments, then there exists a linear dependence among the remaining arguments; i. e. if $[x_1, x_2, \dots, x_{n-1}, y]$ denotes a determinant function in the n -dimensional space and $[x_1, x_2, \dots, x_{n-1}, y] = 0$ for all y , then $x_1, x_2, \dots, x_{n-1}$ are linearly dependent.

Proof. Assume the contrary. Then we can always find a vector x_n such that $x_1, x_2, \dots, x_{n-1}, x_n$ are linearly independent. Hence by the corollary to Theorem 1, the determinant function $[x_1, x_2, \dots, x_{n-1}, x_n]$ is different from zero. But by hypothesis $[x_1, x_2, \dots, x_{n-1}, y] = 0$ for all y ; hence for $y = x_n$, $[x_1, x_2, \dots, x_{n-1}, x_n] = 0$, which is a contradiction. Therefore, the theorem.

Theorem 4. For any linear scalar function of a vector

variable in a linear vector space of dimension n , there exists an $(n-1)$ -dimensional subspace on which the function vanishes.

Proof. Let $f(x)$ denote the function in question. If $f(x)$ vanishes for all x , the theorem is self-evident. Suppose for some x_1 , $f(x_1) \neq 0$. We can always find $x_2, x_3, \dots, x_n$ such that $x_1, x_2, \dots, x_n$ are linearly independent. Let $x'_2 = x_2 - \frac{f(x_2)}{f(x_1)}x_1$, $x'_3 = x_3 - \frac{f(x_3)}{f(x_1)}x_1, \dots, x'_n = x_n - \frac{f(x_n)}{f(x_1)}x_1$

where $f(x_i), i = 2, 3, \dots, n$, may or may not be zero. Then $x_1, x'_2, x'_3, \dots, x'_n$ are also linearly independent, and $f(x'_i) = 0$ for $i = 2, 3, \dots, n$. So any vector x in the subspace formed by linear combinations of $x'_2, x'_3, \dots, x'_n$ will make $f(x)$ vanish, since we can write $x = \xi_2 x'_2 + \xi_3 x'_3 + \dots + \xi_n x'_n$ and $f(x) = f(\xi_2 x'_2 + \xi_3 x'_3 + \dots + \xi_n x'_n) = \xi_2 f(x'_2) + \xi_3 f(x'_3) + \dots + \xi_n f(x'_n) = 0$

In other words, the function vanishes on this subspace of dimension $n-1$.

The *zero space* of a linear vector function on a linear vector space of dimension n is the subspace on which the function vanishes. The *value space* of the same is the subspace formed by all vector values of the function. Using this terminology, we have

Theorem 5. If the zero space of a linear vector function on a linear vector space of dimension n has dimension p and its value space has dimension q , then $p+q=n$.

Proof. By hypothesis, there exist in the zero space p linearly independent vectors, $x_1, x_2, \dots, x_p$. We can always find $x_{p+1}, x_{p+2}, \dots, x_n$ such that $x_1, x_2, \dots, x_n$ are linearly independent. Then for any x in the general space we can write

$$x = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n,$$

ξ being scalars. And if $f(x)$ denotes the linear vector function, $f(x_i) = 0$ for $i = 1, 2, \dots, p$.

Hence

$$f(x) = f(\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n) = \xi_{p+1} f(x_{p+1}) + \xi_{p+2} f(x_{p+2}) + \dots + \xi_n f(x_n)$$

for all x of the space, i. e. all vector values of the function f can be expressed by linear combinations of at most $n-p$ linearly independent vectors, namely

$$f(x_{p+1}), f(x_{p+2}), \dots, f(x_n)$$

In other words, the value space of f must have dimension $q \leq n-p$. Suppose $f(x_{p+1}), f(x_{p+2}), \dots, f(x_n)$, are not linearly independent; then there exist $\eta_{p+1}, \eta_{p+2}, \dots, \eta_n$, not all zero, such that

$$\eta_{p+1} f(x_{p+1}) + \eta_{p+2} f(x_{p+2}) + \dots + \eta_n f(x_n) = 0$$

By the linear property of f , we have

$$f(\eta_{p+1} x_{p+1} + \eta_{p+2} x_{p+2} + \dots + \eta_n x_n) = 0$$

hence the vector

$$x'_{p+1} = \eta_{p+1} x_{p+1} + \eta_{p+2} x_{p+2} + \dots + \eta_n x_n \neq 0$$

is in the zero space of f and should be linearly dependent on $x_1, x_2, \dots, x_p$. But since $x_1, x_2, \dots, x_p, x_{p+1}, \dots, x_n$ are linearly independent, this is impossible. Hence

$$f(x_{p+1}), f(x_{p+2}), \dots, f(x_n)$$

must be linearly independent and consequently $q = n-p$, or $p+q=n$.

Theorem 6. If a symmetric bilinear scalar function on a linear vector space of dimension n , say $s(x, y)$, has the property that with any vector in the space as one of its arguments it has a zero space, with respect to the other argument, of dimension exactly $n-1$, in other words no x of the space can make $s(x, y)$ vanish for all y , then there exists a skew linear vector function of $n-1$ vectors, $x_1, x_2, \dots, x_{n-1}$, denoted by $[x_1, x_2, \dots, x_{n-1}]$, such that $s([x_1, x_2, \dots, x_{n-1}], x_n)$ is a determinant function and

$$\begin{aligned}
& [x'_1, x'_2, \dots, x'_{n-2}, [x_1, x_2, \dots, x_{n-1}]] \\
& = k \begin{vmatrix} s(x'_1, x_1) & s(x'_1, x_2) & s(x'_1, x_3) & \dots & s(x'_1, x_{n-1}) \\ s(x'_2, x_1) & s(x'_2, x_2) & s(x'_2, x_3) & \dots & s(x'_2, x_{n-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s(x'_{n-2}, x_1) & s(x'_{n-2}, x_2) & s(x'_{n-2}, x_3) & \dots & s(x'_{n-2}, x_{n-1}) \end{vmatrix} \\
& \qquad \qquad \qquad x_1 \qquad \qquad x_2 \qquad \qquad x_3 \qquad \qquad x_{n-1}
\end{aligned}$$

k being a scalar constant ¹

Proof. There must exist in the space a vector u_1 such that

$$s(u_1, u_1) \neq 0, \text{ since if } s(x, x) = 0 \text{ for all } x$$

$$\text{then } s(x, y) = \frac{s(x+y, x+y) - s(x, x) - s(y, y)}{2} = 0$$

for any x, y in the space, contradicting the hypothesis. By Theorem 4, $s(u_1, x)$ as a function of x has an $(n-1)$ -dimensional zero space which does not contain u_1 , and which will be denoted by π_1^{n-1} . In π_1^{n-1} there must exist a u_2 such that $s(u_2, u_2) \neq 0$, since if $s(x, x) = 0$ for all x in π_1^{n-1} then $s(x, y) = 0$ for all x, y in π_1^{n-1} and also $s(x, u_1) = s(u_1, x) = 0$ as x is in the zero space of $s(u_1, x)$, hence $s(x, y) = 0$ for all y in the entire space, contradictory to hypothesis. Similarly $s(u_2, x)$ as a function of x has an $(n-1)$ -dimensional zero space π_2^{n-1} which does not contain u_2 . Now π_2^{n-1} must contain u_1 as $s(u_2, u_1) = 0$, while π_1^{n-1} does not contain u_1 ; hence $\pi_1^{n-1} \neq \pi_2^{n-1}$. Yet they must have an $(n-2)$ -dimensional subspace in common, otherwise there would be more than n linearly independent vectors in the n -dimensional space. Denote that common subspace by π_{12}^{n-2} . Then π_{12}^{n-2} does not contain u_1 and u_2 . Using same argument as before we can find in π_{12}^{n-2} a u_3 such that $s(u_3, u_3) \neq 0$. For $s(u_3, x)$, there is a zero space π_3^{n-1} , and π_1^{n-1} , π_2^{n-1} , π_3^{n-1} must have an $(n-3)$ -dimensional subspace π_{123}^{n-3} in common which does not contain u_1, u_2 , and u_3 . In π_{123}^{n-3} we can find a u_4 such that $s(u_4, u_4) \neq 0$. Proceeding in this way, we will finally obtain an orthogonal system² $u_1, u_2, \dots,$

u_n such that $s(u_i, u_j) = 0$ for $i \neq j$, $i, j = 1, 2, \dots, n$, and $s(u_i, u_i) \neq 0$; besides, $u_1, u_2, \dots, u_n$ are linearly independent, for otherwise we would be able to write $u_n = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_{n-1} u_{n-1}$ and for $i \neq n$,

$$s(u_i, u_n) = \alpha_i s(u_i, u_i)$$

where at least one $\alpha_i, 1 \leq i \leq n-1$, must not be zero, yet $s(u_i, u_n) = 0$ for $i = 1, 2, \dots, n-1$ and $s(u_i, u_i) \neq 0$. Then for any vector x we can write $x = x^1 u_1 + x^2 u_2 + \dots + x^n u_n$ where $x^i, i = 1, 2, \dots, n$, are scalars. Let $x_1, x_2, \dots, x_{n-1}$ denote $n-1$ variable vectors, where $x_i = x_i^1 u_1 + x_i^2 u_2 + \dots + x_i^n u_n, i = 1, 2, \dots, n-1$. Then they are linearly dependent if and only if all the determinants

$$\begin{vmatrix} x_1^1 x_1^2 \dots x_1^{i-1} x_1^{i+1} \dots x_1^n \\ x_2^1 x_2^2 \dots x_2^{i-1} x_2^{i+1} \dots x_2^n \\ \vdots \\ x_{n-1}^1 x_{n-1}^2 \dots x_{n-1}^{i-1} x_{n-1}^{i+1} \dots x_{n-1}^n \end{vmatrix} = 0, \quad i = 1, 2, \dots, n.$$

Therefore if we form $D(x_1, x_2, \dots, x_{n-1})$

$$\begin{aligned} & \begin{vmatrix} s(x_1, u_1) s(x_1, u_2) \dots s(x_1, u_n) \\ s(x_2, u_1) s(x_2, u_2) \dots s(x_2, u_n) \\ \vdots \\ s(x_{n-1}, u_1) s(x_{n-1}, u_2) \dots s(x_{n-1}, u_n) \\ u_1 \quad u_2 \quad \dots \quad u_n \end{vmatrix} \\ &= \begin{vmatrix} x_1^1 s(u_1, u_1) \quad x_1^2 s(u_2, u_2) \dots x_1^n s(u_n, u_n) \\ x_2^1 s(u_1, u_1) \quad x_2^2 s(u_2, u_2) \dots x_2^n s(u_n, u_n) \\ \vdots \\ x_{n-1}^1 s(u_1, u_1) \quad x_{n-1}^2 s(u_2, u_2) \dots x_{n-1}^n s(u_n, u_n) \\ u_1 \quad u_2 \quad \dots \quad u_n \end{vmatrix} \\ &= \sum_{i=1}^n (-1)^{i+1} \begin{vmatrix} x_1^1 x_1^2 \dots x_1^{i-1} x_1^{i+1} \dots x_1^n \\ x_2^1 x_2^2 \dots x_2^{i-1} x_2^{i+1} \dots x_2^n \\ \vdots \\ x_{n-1}^1 x_{n-1}^2 \dots x_{n-1}^{i-1} x_{n-1}^{i+1} \dots x_{n-1}^n \end{vmatrix} s(u_1, u_1) s(u_2, u_2) \dots s(u_n, u_n) u_i \end{aligned}$$

We see that $D(x_1, x_2, \dots, x_{n-1})$ vanishes if and only if $x_1, x_2, \dots, x_{n-1}$ are linearly dependent, and

$$s(D(x_1, x_2, \dots, x_{n-1}), x) = 0 \quad \text{for } i=1, 2, \dots, n-1.$$

Also $D(x_1, x_2, \dots, x_{n-1})$ is linear in $x_1, x_2, \dots, x_{n-1}$.

Hence for any other vector x_n ,

$$\begin{aligned} & s(D(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{n-1}), x_n) \\ &= \begin{vmatrix} x_1^1 s(u_1, u_1) \dots x_n^1 s(u_n, u_n) \\ x_1^2 s(u_1, u_1) \dots x_n^2 s(u_n, u_n) \\ \vdots \\ x_1^{n-1} s(u_1, u_1) \dots x_n^{n-1} s(u_n, u_n) \\ s(u_1, x_n) \dots s(u_n, x_n) \end{vmatrix} \\ &= \begin{vmatrix} x_1^1 s(u_1, u_1) \dots x_1^n s(u_n, u_n) \\ x_2^1 s(u_1, u_1) \dots x_2^n s(u_n, u_n) \\ \vdots \\ x_{n-1}^1 s(u_1, u_1) \dots x_{n-1}^n s(u_n, u_n) \\ x_1^n s(u_1, u_1) \dots x_n^n s(u_n, u_n) \end{vmatrix} \\ &= s(u_1, u_1) s(u_2, u_2) \dots s(u_n, u_n) \begin{vmatrix} x_1^1 & x_2^1 & \dots & x_n^1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \\ x_1^n & x_2^n & \dots & x_n^n \end{vmatrix} \end{aligned}$$

Since $s(u_1, u_1) s(u_2, u_2) \dots s(u_n, u_n)$ is a scalar constant not equal to zero, we see that $s(D(x_1, x_2, \dots, x_{n-1}), x_n)$ is a determinant function. Therefore the existence of $[x_1, x_2, \dots, x_{n-1}]$, as the theorem states, is realized by $D(x_1, x_2, \dots, x_{n-1})$. Now we can denote $D(x_1, x_2, \dots, x_{n-1})$ by $[x_1, x_2, \dots, x_{n-1}]$. To prove the second part of the Theorem, we let

$$[x'_1, x'_2, \dots, x'_{n-2}, [x_1, x_2, \dots, x_{n-1}]] = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_{n-1} x_{n-1} + \xi_n z$$

where z is a vector linearly independent of $x_1, x_2, \dots, x_{n-1}$. If

$$\begin{aligned} &S([x'_1, x'_2, \dots, x'_{n-2}, [x_1, x_2, \dots, x_{n-1}]], [x_1, x_2, \dots, x_{n-1}]) = 0 \\ &S(x_i, [x_1, x_2, \dots, x_{n-1}]) = 0, \quad i = 1, 2, \dots, n-1 \end{aligned}$$

So we have

Also

And

[illegible]

Or

Theorem 7. In a 3-dimensional linear vector space, the most

general form of a skew bilinear scalar function of two vectors, say x, y , is a determinant function one of those arguments is an arbitrary constant vector, such as $[k, x, y]$, k being a constant vector.

Proof. Let $f(x, y)$ denote any skew bilinear scalar function of x, y . Then $f(x, x) = 0$ for any x . There must exist x_0 and y_0 such that $f(x_0, y_0) \neq 0$; otherwise the function would be a constant 0 and $f(x, y) = [0, x, y]$. By Theorem 4, $f(x_0, y)$ as a function of y has a 2-dimensional zero space, the plane π_{x_0} . Likewise $f(x, y_0)$ as a function of x has a zero space π_{y_0} . These two planes, π_{x_0} and π_{y_0} , must have a direction in common. Take a vector K' in that direction; then

$$f(x_0, k') = f(k', y_0) = 0$$

We see that x_0, y_0 , and k' must be linearly independent; otherwise $f(x_0, y_0)$ would vanish. Let $f(x_0, y_0) = \iota [k', x_0, y_0]$ where ι is a scalar constant. Then, since we can always write for any x, y in the space, $x = x^1 x_0 + x^2 y_0 + x^3 k'$ and $y = y^1 x_0 + y^2 y_0 + y^3 k'$ where x^i and y^i are scalars, we have

$$\begin{aligned} f(x, y) &= f(x^1 x_0 + x^2 y_0 + x^3 k', y^1 x_0 + y^2 y_0 + y^3 k') = (x^1 y^2 - x^2 y^1) f(x_0, y_0) \\ &= (x^1 y^2 - x^2 y^1) \iota [k', x_0, y_0] = \iota [k', x, y], \end{aligned}$$

or if we let $\iota k' = k$, we have $f(x, y) = [k, x, y]$ for all x and y .

Theorem 8. In a 4-dimensional linear vector space, the most general form of a skew bilinear scalar function of two vectors, say x and y , is the sum of two determinant functions in each of which two of its arguments are arbitrary constant vectors, such as $[a, b, x, y] + [c, d, x, y]$, where a, b, c, d are constant vectors.

Proof. Let again $f(x, y)$ denote any skew bilinear scalar function of x and y . Then $f(x, x) = 0$ for all x in the space. If $f(x, y) = 0$ for all x and y , then the theorem is trivial, with $a = b = c = d = 0$, $f(x, y)$ being always possible to be expressed as $[a, b, x, y] + [c, d, x, y]$. Suppose there exist a' and b' such

that $f(a', b') \neq 0$. Then for $f(a', y)$ as a function of y there is a 3-dimensional zero space $\pi a'$, by Theorem 4. which does not contain b' . Similarly for $f(x, b')$ as function of x there is a 3-dimensional zero space $\pi b'$ not containing a' . There will be a plane common to $\pi a'$ and $\pi b'$, which does not contain a' and b' . Take two non-collinear vectors c' and d' in that plane. Then $f(a', c') = f(a', d') = 0$ and $f(c', b') = f(d', b') = 0$, and a', b', c', d' must be linearly independent otherwise $f(a', b')$ would vanish. Hence we can write for any x and y in the space $x = x^1 a' + x^2 b' + x^3 c' + x^4 d'$ and $y = y^1 a' + y^2 b' + y^3 c' + y^4 d'$, and $f(x, y) = (x^1 y^2 - x^2 y^1) f(a', b') + (x^3 y^4 - x^4 y^3) f(c', d')$. Let $f(a', b') = \iota [c', d', a', b']$, $f(c', d') = m [a', b', c', d']$ where ι and m are scalar constants and $m = 0$ if $f(c', d') = 0$. Then $f(x, y) = \iota [c', d', xy] + m [a', b', xy]$ or by absorbing ι and m , a', b', c', d' become a, b, c, d , and $f(x, y) = [a, b, x, y] + [c, d, x, y]$.

1. Compare J. B. Shaw's Synopsis of Linear Associative Algebras, Part 1, Chapt. I, 11, 12, 13, pp. 11—12.
2. For definition of an orthogonal system see J. B. Shaw's Synopsis of Linear Associative Algebras, Part 1, Chapt. I, §9, pp. 11; or J. V. Neumaun's Functional Operators (Notes by Dr. R. S. Martin, 1934—35), p. 2—8a, Definition 12.8.

§3 THE NECESSARY AND SUFFICIENT CONDITIONS ON θ AND ϕ FOR AN ASSOCIATIVE σ .

By definition, we have

$$\theta(x,y) = \frac{\sigma(x,y) + \sigma(y,x)}{2}, \quad \phi(x,y) = \frac{\sigma(x,y) - \sigma(y,x)}{2}$$

The associative property gives $\sigma(\sigma(x,y),z) = \sigma(x,\sigma(y,z))$

By expressing the σ -functions in terms of θ - and ϕ - functions, we have

$$\begin{aligned} \sigma(\theta(x,y) + \phi(x,y), z) &= \sigma(\theta(x,y), z) + \sigma(\phi(x,y), z) \\ &= [\theta(\theta(x,y), z) + \phi(\theta(x,y), z)] + [\theta(\phi(x,y), z) + \phi(\phi(x,y), z)] \\ &= \sigma(x, \theta(y,z) + \phi(y,z)) = \sigma(x, \theta(y,z)) + \sigma(x, \phi(y,z)) \\ &= [\theta(x, \theta(y,z)) + \phi(x, \theta(y,z))] + [\theta(x, \phi(y,z)) + \phi(x, \phi(y,z))] \end{aligned}$$

Hence

$$\begin{aligned} [3.1] \quad &\theta(\theta(x,y), z) + \theta(\phi(x,y), z) + \phi(\theta(x,y), z) + \phi(\phi(x,y), z) \\ &= \theta(x, \theta(y,z)) + \theta(x, \phi(y,z)) + \phi(x, \theta(y,z)) + \phi(x, \phi(y,z)) \end{aligned}$$

Likewise

$$\begin{aligned} [3.2] \quad &\theta(\theta(y,z), x) + \theta(\phi(y,z), x) + \phi(\theta(y,z), x) + \phi(\phi(y,z), x) \\ &= \theta(y, \theta(z,x)) + \theta(y, \phi(z,x)) + \phi(y, \theta(z,x)) + \phi(y, \phi(z,x)) \end{aligned}$$

$$\begin{aligned} \text{and } [3.3] \quad &\theta(\theta(z,x), y) + \theta(\phi(z,x), y) + \phi(\theta(z,x), y) + \phi(\phi(z,x), y) \\ &= \theta(z, \theta(x,y)) + \theta(z, \phi(x,y)) + \phi(z, \theta(x,y)) + \phi(z, \phi(x,y)) \end{aligned}$$

By adding the right sides of [3.1], [3.2], [3.3] together, and their left sides together, and cancelling similar terms, we get

$$\begin{aligned} &\phi(\theta(x,y), z) + \phi(\theta(y,z), x) + \phi(\theta(z,x), y) + \phi(\phi(x,y), z) \\ &\quad + \phi(\phi(y,z), x) + \phi(\phi(z,x), y) = 0 \end{aligned}$$

Now, [J] $\phi(\phi(x,y), z) + \phi(\phi(y,z), x) + \phi(\phi(z,x), y) = 0$ known as the Jacobian Identity; for

$$\phi\left(\frac{\sigma(x,y)-\sigma(y,x)}{2}, z\right) + \phi\left(\frac{\sigma(y,z)-\sigma(z,y)}{2}, x\right) \\ + \phi\left(\frac{\sigma(z,x)-\sigma(x,z)}{2}, y\right)$$

$$= \frac{1}{4} \{ \sigma(\sigma(x,y), z) - \sigma(z, \sigma(x,y)) - \sigma(\sigma(y,x), z) + \sigma(z, \sigma(y,x)) \\ + \sigma(\sigma(y,z), x) - \sigma(x, \sigma(y,z)) - \sigma(\sigma(z,y), x) + \sigma(x, \sigma(z,y)) \\ + \sigma(\sigma(z,x), y) - \sigma(y, \sigma(z,x)) - \sigma(\sigma(z,y), y) + \sigma(y, \sigma(z,y)) \} = 0$$

due to the associative property of the σ -functions. Hence we have [3.4] $\phi(\theta(x,y), z) + \phi(\theta(y,z), x) + \phi(\theta(z,x), y) = 0$.

Interchanging y and z in [3.2], we have θ

$$\theta(\theta_{z,y}, x) + \theta(\phi(z,y), x) + \phi(\theta(z,y), x) + \phi(\phi(z,y), x) \\ = \theta(z, \theta(y,x)) + \theta(z, \phi(y,x)) + (\phi z, \theta(y,x)) + \phi(z, \phi(y,x))$$

Eliminating common terms of this equation and [3.1], we get

$$\theta(\phi(x,y), z) - \theta(\phi(y,z), x) = -\phi(\theta(x,y), z) - \phi(\theta(y,z), x)$$

and consequently

$$\theta(\theta(x,y), z) - \theta(\theta(y,z), x) = -\phi(\phi(x,y), z) - \phi(\phi(y,z), x).$$

Using [3.4] and the Jacobian Identity, we have

$$[3.5] \quad \theta(\phi(x,y), z) - \theta(\phi(y,z), x) = \phi(\theta(z,x), y), \quad \text{and}$$

$$[3.6] \quad \theta(\theta(x,y), z) - \theta(\theta(y,z), x) = \phi(\phi(z,x), y),$$

which are the necessary and sufficient conditions for an associative σ -operation.

§4. ENTIRELY NON-COMMUTATIVE SPACE

In any algebra, the product of two vectors of the same direction is commutative. If this is the only case where the product is commutative and hence the ϕ -function vanishes, then we call the algebra an entirely non-commutative algebra and the space an entirely non-commutative space; i.e. in this algebra, $\sigma(x,y)=\sigma(y,x)$ if and only if x and y are in the same direction, where x and y denote non-zero vectors in the space. In terms of the skew part $\phi(x,y)$ of the algebra, this property will be expressed as follows: in an entirely non-commutative space, $\phi(x,y)$ vanishes if and only if x and y are collinear. Set $y=z=x$ in [3.5], then $\phi(\theta(x,x),x)=0$, $\therefore \theta(x,x)=2\lambda_x x$ for all x in the space. Set $y=z$ in [3.4], then $2\phi(\theta(x,z),x)+\phi(\theta(z,z),x)=0$; now $\theta(z,z)=2\lambda_z z$. $\therefore \phi(\theta(x,z)-\lambda_z^2 x,z)=0$, $\therefore \theta(x,z)-\lambda_z x=kz$, or $\theta(x,z)=\lambda_z x+kz$; since $\theta(x,z)$ is symmetric in x and z , we easily see that the constant k must be λ_x . Therefore, for any two vectors in the space, say x and z , $\theta(x,z)=\lambda_z x+\lambda_x z$ where λ_z and λ_x are two scalars depending on z and x respectively. Let ι and m be any two scalars. Then $\theta(x,\iota y+mz)=\lambda_{\iota y+mz} x+\lambda_x(\iota y+mz)$, but $\theta(x,\iota y+mz)=\iota\theta(x,y)+m\theta(x,z)=\iota(\lambda_y x+\lambda_x y)+m(\lambda_z x+\lambda_x z)$, $\lambda_{\iota y+mz} x=\iota\lambda_y x+m\lambda_z x$ for any y and z . Therefore λ_z depends on z linearly, and same with λ_y, λ_x , etc. Therefore we have the following theorem:

Theorem 9. The general form of the symmetric part $\theta(x,y)$ of an entirely non-commutative algebra is

[4.1] $\theta(x, y) = \lambda_y x + \lambda_x y$, where λ_y is a linear scalar function of y .

Suppose the space is of dimension n . For $n=1$ we have a commutative algebra. We shall start with $n=2$.

Case I, $n=2$. Take any two linearly independent vectors in the space, say u_1 and u_2 . Then for any vectors $x, y, \dots$ in the space, $x = x^1 u_1 + x^2 u_2, y = y^1 u_1 + y^2 u_2$, etc., where x, y are components. And for any x and y , we have

[4.2] $\phi(x, y) = \phi(x^1 u_1 + x^2 u_2, y^1 u_1 + y^2 u_2) = (x^1 y^2 - x^2 y^1) \phi(u_1, u_2) = [x, y] a$ where $[x, y]$ is a determinant function and a is a fixed direction. Applying [4.1] and [4.2] to [3.6], we obtain $\lambda_x \lambda_y z - \lambda_z \lambda_x = [z, x] [a, y] a = [a y] ([a x] z + [z a] x)$, using Theorem 2, $\therefore \lambda_x = \pm [a x], \lambda_y = \pm [a y]$, etc. Therefore, for any x and y in the space, [4.3] $\theta(x, y) = \pm [a y] x \pm [a x] y$. And [4.4] $\sigma(x, y) = \theta(x, y) + \phi(x, y) = \pm [a y] x \pm [a x] y + [x y] a =$

$2[a y] x$ where $a = \frac{\phi(x, y)}{[x y]}$. Thus we have an associative algebra.

Case II, $n \geq 3$. In $\theta(x, y) = \lambda_y x + \lambda_x y$, λ_x and λ_y are linear scalar functions of the vector variables x and y respectively. They can not vanish for all x and y in the space; for if they do, then for any x, y , and z , $\theta(x, y) = 0$, and $\theta(y, z) = 0$, and [3.6] becomes $0 = \phi(\phi(z, x), y)$ for all x, y , and z , which is impossible since we can always choose z and x such that $\phi(z, x) \neq 0$ and there is always some y not collinear with $\phi(z, x)$ and hence $\phi(\phi(z, x), y) \neq 0$ by hypothesis. By Theorem 4, there exist $n-1$ (≥ 2) linearly independent vectors, say z and x , such that $\lambda_z = 0, \lambda_x = 0$ and consequently $\lambda_t z + m x = 0$ where t and m are any scalars. Let $y = t z + m x$. Then [3.6] becomes again $0 = \phi(\phi(z, x), y)$. Now z and x are linearly independent; $\therefore \phi(z, x) \neq 0$. By varying t and m we can always choose $y = t z + m x$ not collinear with $\phi(z, x)$. Thus we get by hypothesis $\phi(\phi(z, x), y) \neq 0$, a contradiction. That means there is no

possible solution for the form of θ -functions and σ -functions. So we have

Theorem 10. In a space of dimension $n \geq 3$ there are no associative entirely non-commutative algebras. (A special case of Theorem 21, §9)

§ 5. ALGEBRAS IN CENTRALIZED SPACES

Suppose the n -dimensional linear vector space has an r -dimensional linear subspace as a centrum¹ such that for any two vectors x and y in the space $\phi(x, y)$ vanishes if and only if x and y are coplanar with any vector in the centrum. We shall call such one a *centralized space*. Then the equation $\phi(\theta(x, x), x) = 0$ obtained by setting $y = z = x$ in [3.6] requires $\theta(x, x) = 2\lambda_x x + k_{xx}$ where λ_x is a scalar depending on x and k_{xx} is a vector in the centrum also depending on x . Set $z = y$ in [3.4], giving $2\phi(\theta(x, y), y) + \phi(\theta(y, y), x) = 0$; now that $\theta(y, y) = 2\lambda_y y + k_{yy}$, we have $2\phi(\theta(x, y), y) + \phi(2\lambda_y y + k_{yy}, x) = 0$; since k_{yy} is in the centrum, $\therefore \phi(k_{yy}, x) = 0$, $\phi(2\theta(x, y) - 2\lambda_y x, y) = 0$ which requires $\theta(x, y) - \lambda_y x = my + k_{x, y}$, or $\theta(x, y) = \lambda_y x + my + k_{x, y}$. But $\theta(x, y)$ is symmetric in x and y , so m must be λ_x . By the linear and symmetric property of $\theta(x, y)$ we can easily show through a procedure analogous to the one used in §4, that λ_x is a linear scalar function of x and $k_{x, y}$ is a symmetric bilinear function of x and y , for all x and y in the space. So we have

Theorem 11. The general form of the symmetric part $\theta(x, y)$ of an associative algebra in centralized space is

[5.1] $\theta(x, y) = \lambda_y x + \lambda_x y + k_{x, y}$ where λ_x is a linear scalar function of x and $k_{x, y}$ is a symmetric bilinear vector function on the entire space to the centrum which has dimension r .

Using [5.1] in [3.5], we get

$$\lambda_z \phi(x, y) + \lambda_{\phi(x, y)} z + k_{\phi(x, y), z} - \lambda_x \phi(y, z) - \lambda_{\phi(y, z)} x - k_{\phi(y, z), x}$$

$$= \lambda_z \phi(x, y) + \phi(z, y)$$

or, $\lambda \phi(x, y)z - \lambda \phi(y, z)x + k \phi(x, y)_{,z} - k \phi(y, z)_{,x} = 0$. If $r \geq n-1$, we have a commutative space, hence, $\phi(x, y) = 0$ for all x, y ; and for $r \leq n-1$ we can always find z and x outside the centrum so that z, x , and the K 's are linearly independent, hence the coefficients of z and x in the equation must vanish, or [5.2] $\lambda \phi(x, y) = 0$, and $k \phi(x, y)_{,z} = k \phi(y, z)_{,x} = k \phi(z, x)_{,y}$ for all x, y, z . As a consequence, $k \phi(x, y)_{,y} = 0$ and $k \phi(x, y)_{,k} = 0$ for any vector k in the centrum.

Using [5.1] in [3.6], we get

$$\begin{aligned} & \lambda_y (\lambda_z x + \lambda_x z + k_{x,z}) + \lambda_x (\lambda_z y + \lambda_y z + k_{y,z}) + \lambda_z k_{x,y} \\ & + \lambda_{k_{x,y}} z + k_{k_{x,y},z} - \{ \lambda_y (\lambda_x z + \lambda_z x + k_{x,z}) + \lambda_z (\lambda_x y + \lambda_y x + k_{y,x}) \\ & + \lambda_x k_{z,y} + \lambda_{k_{z,y}} x + k_{k_{z,y},x} \} = \phi(\phi(z, x), y) \end{aligned}$$

or

$$\begin{aligned} [5.3] \quad & (\lambda_x \lambda_y + \lambda_{k_{x,y}})z - (\lambda_z \lambda_y + \lambda_{k_{z,y}})x + k_{k_{x,y},z} - k_{k_{z,y},x} \\ & = \phi(\phi(z, x), y) \end{aligned}$$

Theorem 12. There are no associative algebras in a centralized space of dimension $n \geq 2r+3$ where r is the dimension of the centrum.

Proof. Assume the contrary. Then the relation [5.3] will hold for any x, y , and z . Replacing y by $\phi(y, w)$ in [5.3], we have

$$\begin{aligned} & (\lambda_x \lambda \phi(y, w) + \lambda_{k_{x, \phi(y, w)}})z - (\lambda_z \lambda \phi(y, w) + \lambda_{k_{z, \phi(y, w)}})x \\ & + k_{k_{x, \phi(y, w)}, z} - k_{k_{z, \phi(y, w)}, x} = \phi(\phi(z, x), \phi(y, w)) \end{aligned}$$

Or, as [5.2] gives $\lambda \phi(y, w) = 0$, we have

$$\begin{aligned} [5.4] \quad & \lambda_{k_{x, \phi(y, w)}} z - \lambda_{k_{z, \phi(y, w)}} x + k_{k_{x, \phi(y, w)}, z} - k_{k_{z, \phi(y, w)}, x} \\ & = \phi(\phi(z, x), \phi(y, w)); \end{aligned}$$

By interchanging z, x , and y, w , we have

$$\lambda_{k_w, \phi(z, x)}^y - \lambda_{k_y, \phi(z, x)}^w + \lambda_{k_w, \phi(z, x)}^y - \lambda_{k_y, \phi(z, x)}^w \\ = \phi(\phi(y, w), \phi(z, x))$$

$$\therefore \lambda_{k_x, \phi(y, w)}^z - \lambda_{k_z, \phi(y, w)}^x + \lambda_{k_x, \phi(y, w)}^z - \lambda_{k_z, \phi(y, w)}^x \\ = -\lambda_{k_w, \phi(z, x)}^y + \lambda_{k_y, \phi(z, x)}^w - \lambda_{k_w, \phi(z, x)}^y + \lambda_{k_y, \phi(z, x)}^w$$

For $r=0$, the theorem reduces to Case II, 4. For $r>0$, we can always have at least $r+3$ linearly independent vectors outside the centrum and $r+3 \geq 4$. Hence we can pick z, x, y , and w such that they and the K 's are linearly independent. That we must have

$$[5.5] \lambda_{k_x, \phi(y, w)} = \lambda_{k_z, \phi(y, w)} = \lambda_{k_w, \phi(z, x)} = \lambda_{k_y, \phi(z, x)} = 0,$$

which shows that for any three vectors, e. g. x, y, z , such that they and vectors of the centrum are linearly independent.

$\lambda_{k_x, \phi(y, z)} = 0$. From [5.3] we see that $\phi(\phi(z, x), y)$ can be expressed by linear combination of z, x , and vectors of the centrum, where none of their coefficients should vanish for all y since $\phi(z, x)$ is a vector outside centrum with z and x linearly independent of the centrum and by theorem 5 $\phi(\phi(z, x), y)$ as a function of y should have a value space of dimension $\geq r+2$. Therefore if w, z, x and vectors of the centrum are linearly independent, $w, z, \phi(\phi(z, x), y)$, and vectors of the centrum are linearly independent for some y , and so are $w, x, \phi(\phi(z, x), y)$ and vectors of the centrum: so, by

$$[5.5], \lambda_{k_z, \phi(\phi(\phi(z, x), y), w)} = 0. \quad \text{and}$$

$$\lambda_{k_x, \phi(\phi(\phi(z, x), y), w)} = 0, \quad \text{and consequently from [5.4]} \\ \phi(\phi(z, x), \phi(\phi(\phi(z, x), y), w))$$

$$= \lambda_{k_w, \phi(\phi(z, x), y)} - \lambda_{k_y, \phi(z, x)}^w + \lambda_{k_w, \phi(z, x)}^y - \lambda_{k_y, \phi(z, x)}^w$$

But by [5'2] $k \phi(\phi(z,x),y), \phi(z,x) = 0$,

and $k_{w, \phi(z,x), \phi(\phi(z,x),y)} = 0$,

$\phi(\phi(zx), \phi(\phi(\phi(z,x),y),w)) = 0$ for all w such w, z, x , and vectors of the centrum are linearly independent. For w linearly dependent on z, x , and vectors of the centrum,

$\phi(\phi(\phi(z,x),y),w) = \epsilon \phi(z,x)$ where ϵ is some scalar; hence in this case also $\phi(\phi(z,x), \phi(\phi(\phi(z,x),y),w)) = 0$. In other words, for all w in the space we have

[5.6] $\phi(\phi(z,x), \phi(\phi(\phi(z,x),y),w)) = 0$. Since the space has a centrum of dimension r [5.6] requires $\phi(\phi(\phi(z,x),y),w)$ to range, for all w , in not more than $r+1$ dimensions, where $\phi(z,x)$ is not in the centrum for certain z and x on account of Theorem 5. But since $\phi(\phi(z,x),y)$ is not in the centrum, $\phi(\phi(\phi(z,x),y),w)$ has, as a function of w , a zero space of dimension $r+1$, and hence by Theorem 5 it should have a value space of dimension $\geq r+2$ for all w . So we have a contradiction. Therefore the Theorem.

We have no complete solution for the general case of associative algebras in centralized spaces. Special cases will be considered in the following sections. Here we prove two theorems which may be of interest; the first of them. Theorem 13, is used later.

Theorem 13. For an associative algebra in a centralized space of dimension $2r+r$ where $r>0$ is the dimension of the centrum, it is impossible to express all $\phi(x,y)$, x,y ranging through the entire space, in terms of linear combinations of any fixed vector in the space and vectors in the centrum.

Proof. Assume the contrary. Then, for any x and y , $\phi(x,y) = u_{x,y} + c_{x,y}v$ where $u_{x,y}$ is some vector in the centrum and v is a fixed vector outside the centrum with $c_{x,y}$ as its scalar coefficient. Likewise, $\phi(z,w) = u_{z,w} + c_{z,w}v$. And

$$\phi(\phi(z,y),\phi(z,w))=\phi(u_{x,y}+c_{x,y}v,u_{z,w}+c_{z,w}v)=O$$

for any x, y, z, w . By the Jacobian Identity

$$\phi(\phi(x,y),\phi(z,w))+\phi(\phi(y,\phi(z,w)),x)+\phi(\phi(\phi(z,w),x),y)=O$$

$$\text{or } \phi(\phi(y,\phi(z,w)),x)-\phi(\phi(x,\phi(z,w)),y)=O$$

$$\text{or } c_y\phi(z,w)(u_{v,x}+c_{v,x}v)-c_x\phi(z,w)(u_{v,y}+c_{v,y}v)=O$$

$$\text{Hence } \phi(v,c_y\phi(z,w)^x-c_x\phi(z,w)^y)$$

$$=c_y\phi(z,w)\phi(v,x)-c_x\phi(z,w)\phi(v,y)=O$$

for any x, y, z, w . This will be a contradiction if we pick x and y such that x, y, v and vectors of the centrum are linearly independent, unless $c_y\phi(z,w)=O$ for all y ($c_y\phi(z,w)$ vanishes anyway for y linearly dependent on v and vectors of the centrum). But if $c_y\phi(z,w)=O$ for all y , for some z and w such that $\phi(z,w)$ is not in the centrum $\phi(y,\phi(z,w))$, as a function of y , would have a value space of at most r dimensions, while by Theorem 5 it should have a value space of $r+1$ dimensions since its zero space has $r+1$ dimensions. Hence we still have a contradiction. That means the theorem must be true.

Theorem 14. Under the hypothesis of Theorem 13, it is impossible to express all $\phi(x,y)$, x, y ranging through the entire space, in terms of any linear combination of x, y and vectors in the centrum.

Proof. Assume the contrary. Then, for any x and y , $\phi(x,y)=u_{x,y}+c^1_{x,y}x+c^2_{x,y}y$ where $u_{x,y}$ is some vector in the centrum and $c^1_{x,y}, c^2_{x,y}$ are scalars. Thus $\phi(\phi(z,x),y)=u_{\phi(z,x),y}+c^1_{\phi(z,x),y}\phi(z,x)+c^2_{\phi(z,x),y}y$. But by [5.3] we see that $\phi(\phi(z,x),y)$ is a linear combination of vectors in the centrum and z and x only. So we must have $c^2_{\phi(z,x),y}=O$ for any z, x , and y . By Theorems 5 and 13 we can always pick out of the value space of the ϕ -function two vectors ϕ_1 and ϕ_2 such that they and the vectors in the

centrum are linearly independent. Then for any y ,
 $\phi(\phi_{1,y}) = u_{\phi_{1,y}} + c^1_{\phi_{1,y}} \phi_1$, $\phi(\phi_{2,y}) = u_{\phi_{2,y}} + c^1_{\phi_{2,y}} \phi_2$, and
 $\phi(\phi_1, \phi_2) = u_{\phi_1, \phi_2}$. Hence the Jacobian Identity for ϕ_1, ϕ_2 , and
 y becomes $\phi(\phi(\phi_{2,y}), \phi_1) + \phi(\phi(y, \phi_1), \phi_2) = 0$ or
 $c^1_{\phi_{2,y}} \phi(\phi_2, \phi_1) - c^1_{\phi_{1,y}} \phi(\phi_1, \phi_2) = 0$. Since ϕ_1 and ϕ_2 linearly
independent together with the centrum, $\phi(\phi_1, \phi_2) \neq 0$, hence
 $c^1_{\phi_{2,y}} + c^1_{\phi_{1,y}} = 0$. Then $\phi(\phi_1 + \phi_{2,y}) = u_{\phi_{1,y}} + u_{\phi_{2,y}}$
 $+ c^1_{\phi_{1,y}}(\phi_1 - \phi_2)$. But by hypothesis,
 $\phi(\phi_1 + \phi_{2,y}) = u_{\phi_1 + \phi_{2,y}} + c^1_{\phi_1 + \phi_{2,y}}(\phi_1 + \phi_2) + c^2_{\phi_1 + \phi_{2,y}} y$,
so, if we choose y linearly independent of ϕ_1 and vectors in
the centrum, we must have $c^1_{\phi_{1,y}} = 0$ for all y , since $(\phi_1 - \phi_2)$,
 $(\phi_1 + \phi_2)$, y , and the u 's are linearly independent for
the specially chosen y and, for y linearly dependent on
 ϕ_1 and the centrum, $c^1_{\phi_{1,y}}$ vanishes anyway. $\phi(\phi_{1,y}) = u_{\phi_{1,y}}$
for all y . But as ϕ_1 is not in the centrum $\phi(\phi_{1,y})$ as a
function of y has a zero space of $r+1$ dimensions and hence
by Theorem 5 it should have a value space $r+1$ -dimensional
whereas $u_{\phi_{1,y}}$ ranges only in r or less dimensions for all y .
So we get a contradiction. Therefore the Theorem.

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1. For definition of a centrum of an algebra, see Albert's Modern Higher Algebra, p. 12, p. 56.

§6. ONE WAY COMMUTATIVE ALGEBRAS IN n-DIMENSIONAL SPACE; SPECIAL CASES: $n=2, n=3$.

Algebras in centralized space with centrum of r dimensions may be called *r-way commutative algebras*. For $r=1$ we have *1-way commutative algebras* which will be considered in this and the following sections. Then $k_{x,y}$ of [5.1] becomes a fixed direction u with variable coefficient depending on x and y , and in place of [5.1] we have [6.1] $\theta(x,y) = \lambda_y x + \lambda_x y + s(x,y) u$ where $s(x,y)$ is a symmetric bilinear scalar function. [5.2] reduces to

$$[6.2] \quad \lambda_{\phi(x,y)} = 0, \quad s(\phi(x,y), z) = s(\phi(y,z), x) = s(\phi(z,x), y), \\ s(\phi(x,y), y) = 0, \text{ and } \phi(\phi(x,y), u) = 0, \text{ for any } x, y, z.$$

And [5.3] is replaced by

$$[6.3] \quad (\lambda_x \lambda_y + s(x,y) \lambda_{u/z} - (\lambda_z \lambda_y + s(z,y) \lambda_u)_x + (s(x,y) s(u,z) - s(z,y) s(u,x)) u = \phi(\phi(z,x), y)$$

If we replace x and y by u in [6.3], the right hand side vanishes, so in the left hand side the coefficients of z and u must both vanish, i.e.

$$[6.4] \quad \lambda_u \lambda_u + s(u,u) \lambda_u = 0, \text{ and } \lambda_z \lambda_u + s(z,u) \lambda_u = 0.$$

By Theorem 12, we see that in spaces of dimension $n \geq 5$ there are no 1-way commutative associative algebras. So we shall consider only three cases, namely, $n=2, n=3, n=4$.

Case I, $n=2$. It is obvious that this algebra is entirely commutative. Hence we can choose u in any direction. The ϕ -function in the algebra is identically zero, hence $\sigma(x,y) = \theta(x,y)$ for any x,y . If there is to be a non-trivial σ -operation, there must exist one direction, which we shall

take as u , such that $\theta(u,u) \neq 0$; for otherwise we would have

$$\sigma(x,y) = \theta(x,y) = \frac{\theta(x+y,x+y) - \theta(x,x) - \theta(y,y)}{2} = 0$$

for all x,y . From [6.4] we see that, if $\lambda_u \neq 0$, $\lambda = -s(u,u)$ and $\theta(u,u) = -s(u,u)u$, and if $\lambda_u = 0$, $\theta(u,u)u$; hence in either case $s(u,u) \neq 0$. By Theorem 4, there exists another direction v , linearly independent of u , such that $s(u,v) = 0$.

1. Suppose $\lambda_u \neq 0$. Then, from [6.4],

$$\lambda_u = -s(u,u), \quad \lambda_v = -s(v,v) = 0.$$

If $s(v,v) = 0$, then the s -function is a non-definite form 1 and by adjusting the length of u we can have

$$s(u,u) = 1, \quad \text{and}$$

$$[6.5] \quad \sigma(x,y) = \theta(x,y) = s(x,y)u + \lambda_y x + \lambda_x y = (x^1 y^1 - 2x^1 y^1)u - (x^2 y^1 + x^1 y^2)v = -x^1 y^1 u - (x^1 y^2 + x^2 y^1)v$$

where $x = x^1 u + x^2 v$, $y = y^1 u + y^2 v$, x^1 and y^1 being scalars.

If $s(v,v) \neq 0$, then, if we adjust the length of v properly and allow ourselves to use the field of complex numbers, we can have $s(v,v) = 1$, and, for $x = x^1 u + x^2 v$ and $y = y^1 u + y^2 v$,

$$[6.6] \quad \sigma(x,y) = \theta(x,y) = (x^1 y^1 + x^2 y^2)u - y^1(x^1 u + x^2 v) - x^1(y^1 u + y^2 v) = -(x^1 y^1 - x^2 y^2)u - (x^1 y^2 + x^2 y^1)v$$

which gives the algebra of complex numbers where $u = -1$ and $v = -\sqrt{-1}$

2. Suppose $\lambda_u = 0$. Then from [6.3] we obtain by setting $z = u, x = y = v$, $\lambda_v \lambda_u + s(v,v)u = 0$. If $s(v,v) = 0$, s -function is non-definite, and $\lambda_v = 0$. Hence

$$[6.7] \quad \sigma(x,y) = \theta(x,y) = s(x,y)u = x^1 y^1 u \quad \text{where } x = x^1 u + x^2 v, \\ y = y^1 u + y^2 v, \text{ and } s(u,u) = 1.$$

If $s(v,v) \neq 0$, then $\lambda_v^2 = -s(v,v)s(u,u)$. Using the field of complex numbers, we can have either $s(u,u) = s(v,v) = i$ and $\lambda_v = \pm 1$, or $s(u,u) = 1, s(v,v) = -1$ (Pseudo-Euclidean)² and $\lambda_v = \pm 1$, or $s(u,u) = s(v,v) = 1$ and $\lambda_v = \pm i, i = \sqrt{-1}$.

Accordingly,

$$\begin{aligned}
[6.8] \quad \sigma(x,y) &= \theta(x,y) = (ix^1y^1 + ix^2y^2 \pm x^1y^2 \pm x^2y^1)u \pm 2x^2y^2v \\
\text{or} \quad &= (x^1y^1 - x^2y^2 \pm x^1y^2 \pm x^2y^1)u \pm 2x^2y^2v \\
\text{or} \quad &= (x^1y^1 + x^2y^2 \pm ix^1y^2 \pm ix^2y^1)u \pm 2ix^2y^2v.
\end{aligned}$$

Case II, $n=3$. With the help of Theorem 7 we can always express $\phi(x,y)$ for any x and y as follows:

$$\phi(x,y) = [b_1xy] a_1 + [b_2xy] a_2 + [b_3xy] a_3$$

Where a_1, a_2 and a_3 are linearly independent vectors and $[x,y,z]$ denotes a determinant function, and $[b_i xy]$, $i=1, 2, 3$, are skew bilinear scalar functions of x and y with b_i the constant vectors. Since the algebra is one way commutative and u the centrum, $\phi(u,y)=0$ for any y . $\therefore [b_1uy] = [b_2uy] = [b_3uy] = 0$ for all y . Hence by Theorem 3 $b_1 = \lambda u$, $b_2 = \mu u$, $b_3 = \nu u$ where λ, μ, ν are some scalars. Then

$$\phi(x,y) = [uxy](\lambda a_1 + \mu a_2 + \nu a_3) = [uxy]a, \text{ for any } x,y.$$

Hence we have

Theorem 15. In 1-way commutative algebras in 3-dimensional space the skew part of the σ -operation has the form

[6.9] $\phi(x,y) = [uxy] a$ where u is the centrum, and a a fixed vector. Therefore in the right hand side of [6.3] we have

$$\phi(\phi(z,x),y) = [uzx] [uay]a = [uay] ([azx]u + [uax]z + [uza]x)$$

by using Theorem 2. Equating the coefficients of u in [6.3], we get

[6.10] $s(x,y)s(u,z) - s(z,y)s(u,x) = [uay] [azx] 1$. If a is in the direction of u , then by [6.2] $\lambda_u = 0$, and $s(u,x) = 0$ for all x in the space. By setting $x=u, y=x$ in [6.3] we have $\lambda_x \lambda_x u = 0$, $\therefore \lambda_x = 0$ for all x . Therefore [6.1] becomes

[6.11] $\theta(x,y) = s(x,y)u$.

If $s(x,y) = 0$ for all x and y , then $\theta(x,y) = 0$ and [6.12] $\sigma(x,y) = \phi(x,y) = [uxy] u$.

If $s(x,y)$ does not vanish for all x,y , then there must exist a direction, say the vector v , such that $s(v,v) \neq 0$. Take any other direction, v' such that u, v, v' are linearly independent. Let $w = v' - \frac{s(v,v')}{s(v,v)}v$, where $s(v,v')$ may or may not

vanish. Then v, v, w are also linearly independent, and $s(v, w) = 0$. For any x and y we can always write $x = x^1 u + x^2 v + x^3 w$ and $y = y^1 u + y^2 v + y^3 w$ where x^i and y^i , $i = 1, 2, 3$ are scalars. If $s(w, w) = 0$, we have $\theta(x, y) = x^2 y^2 s(v, v) u = x^2 y^2 u$ since we can always fix the length of v , so that $s(v, v) = 1$. And

$$[6.13] \quad \sigma(x, y) = x^2 y^2 u + [uxy]u = (x^2 y^2 + (x^2 y^3 - x^3 y^2) [uvw])u \\ = (x^2 y^2 + x^2 y^3 - x^3 y^2)u \text{ if we fix the length of } w \text{ so} \\ \text{that } [uvw] = 1.$$

If $s(w, w) \neq 0$, we can also make $s(w, w) = 1$ using the field of complex numbers. Then $\theta(x, y) = (x^2 y^2 + x^3 y^3)u$, and

$$[6.14] \quad \sigma(x, y) = (x^2 y^2 + x^3 y^3 + (x^2 y^3 - x^3 y^2) [vww])u$$

2. If a and u are non-collinear, then $s(u, x)$ must not vanish for some x , since if it vanishes for all x then the left hand side of [6.10] is identically zero while the right hand side is not. From [6.2] we have $\lambda_a = 0$ and $s(a, x) = 0$ for all x . If $s(u, u) = 0$ then there exists a vector v such that $s(u, v) \neq 0$ and u, v and a are linearly independent. If $s(v, v) = 0$, we let $b = u + v$ so that $s(b, c) = 0$ but $s(b, b) \neq 0$, $s(c, c) \neq 0$, and we fix their lengths so that $s(b, b) = 1$, $s(c, c) = -1$. If $s(v, v) \neq 0$, we let $b = v$ and $c = u - \frac{s(u, v)}{s(v, v)} v$ so that $s(b, c) = 0$, but $s(b, b) \neq 0$, $s(c, c) \neq 0$, and we fix their lengths so that $s(bb) = 1$, $s(c, c) = -1$. In either case we can write for any x, y , $x = x^1 a + x^2 b + x^3 c$ and $y = y^1 a + y^2 b + y^3 c$ with x_i and y_i , $i = 1, 2, 3$, their components, since a, b , and c are linearly independent. Then $s(x, y) = x^2 y^2 - x^3 y^3$ for any x, y , and [6.10] becomes

$$(x^3 z^2 - x^2 z^3)(y^2 u^3 - y^3 u^2) = [uay] [azx] \\ = (y^2 u^3 - y^3 u^2) [abc] (z^2 x^3 - z^3 x^2) [abc] \\ \therefore [abc] = 1, \text{ and } \phi(x, y) = [uxy]a = \pm \{u^2(x^3 y^1 - x^1 y^3) \\ + u^3(x^1 y^2 - x^2 y^1)\}a$$

From [6.4], $\lambda_u = 0$ since $s(u, u) = 0$. Setting $y = x, z = a$ in [6.3] we have

$$\lambda_x \lambda_x a = \phi(\phi(a, x), x) = [uax][nax]a, \therefore \lambda_x = \pm uax = \pm (u^3 x^2 - u^2 x^3)$$

for any x . Hence

$$\begin{aligned}\theta(x, y) &= \lambda_y x + \lambda_x y + s(x, y)u = \pm (u^3 y^2 - u^2 y^3)x \pm (u^3 x^2 - u^2 x^3)y \\ &\quad + (x^2 y^2 - x^3 y^3)u = (\pm (u^3 y^2 - u^2 y^3)x^1 \pm (u^3 x^2 - u^2 x^3)y^1)a \\ &\quad + (\pm (u^3 y^2 - u^2 y^3)x^2 \pm (u^3 x^2 - u^2 x^3)y^2 + (x^2 y^2 - x^3 y^3)u^2)b \\ &\quad + (\pm (u^3 y^2 - u^2 y^3)x^3 \pm (u^3 x^2 - u^2 x^3)y^3 + (x^2 y^2 - x^3 y^3)u^3)c\end{aligned}$$

And

$$\begin{aligned}[6.15] \quad \sigma(x, y) &= (\pm (u^3 y^2 - u^2 y^3)x^1 \pm (u^3 x^2 - u^2 x^3)y^1 \\ &\quad \pm (x^3 y^1 - x^1 y^3)u^2 \pm (x^1 y^2 - x^2 y^1)u^3)a \\ &\quad + (\pm (u^3 y^2 - u^2 y^3)x^2 \pm (u^3 x^2 - u^2 x^3)y^2 + (x^2 y^2 - x^3 y^3)u^2)b \\ &\quad + (\pm (u^3 y^2 - u^2 y^3)x^3 \pm (u^3 x^2 - u^2 x^3)y^3 + (x^2 y^2 - x^3 y^3)u^3)c\end{aligned}$$

If $s(u, u) \neq 0$, then by Theorem 4 there exists a vector v such that $s(u, v) = 0$ and u, v, a are linearly independent. Setting $z = u$, $x = y = v$ in [6.10], we have $s(v, v)s(u, u) = [uav][auv] = -[uva]^2 \neq 0, \therefore (v, v) \neq 0$. We can fix the lengths of u and v so that $s(u, u) = 1$, $s(v, v) = -1$, then $[uva] = \pm 1$. If we write for any x and y $x = x^1 u + x^2 v + x^3 a$ and $y = y^1 u + y^2 v + y^3 a$ with x_i and y_i as their components, we have $\phi(x, y) = [uxy]a = \pm (x^2 y^3 - x^3 y^2)a$ and $s(x, y) = x^1 y^1 - x^2 y^2$. If $\lambda_u = 0$, we have as before for any x

$$\begin{aligned}\lambda_x &= \pm [uax] = \pm x^2, \text{ Hence } \theta(x, y) = \pm y^2 x \pm x^2 y + (x^1 y^1 - x^2 y^2)u, \\ \text{and } [6.16] \quad \sigma(x, y) &= \pm y^2 x \pm x^2 y + (x^1 y^1 - x^2 y^2)u \pm (x^2 y^3 - x^3 y^2)a \\ &= (\pm y^2 x^1 \pm x^2 y^1 + x^1 y^1 - x^2 y^2)u \pm 2x^2 y^2 v \pm x^2 y^3 a\end{aligned}$$

If $\lambda_u \neq 0$, then by [6.4] $\lambda = -s(u, z)$ for any z ,

$$\begin{aligned}\therefore \theta(x, y) &= -s(x, y)x - s(u, x)y + s(x, y)u \\ &= -y^1 x - x^1 y + (x^1 y^1 - x^2 y^2)u = -(x^1 y^1 + x^2 y^2)u - (y^1 x^2 + x^1 y^2)v \\ &\quad - (y^1 x^3 + x^1 y^3)a\end{aligned}$$

$$\begin{aligned}\text{And } [6.17] \quad \sigma(x, y) &= -(x^1 y^1 + x^2 y^2)u - (y^1 x^2 + y^2 x^1)v \\ &\quad - (x^3 y^1 + x^1 y^3 \pm x^2 y^3 \mp x^3 y^2)a\end{aligned}$$

as a conclusion we can formulate that for $n=3$, as soon as the ϕ -function is defined we have in the 1-way commutative algebra two vectors u and a , which may lie in same direction

giving associative operation σ in three forms [6.12], [6.13], and [6.14], where the s -function is a symmetric bilinear form vanishing identically, vanishing in a plane containing a , and being positive definite³ in a plane not containing a respectively, and for u and a not in same direction we have [6.15], [6.16] and [6.17] where the s -function is defined in a pseudo-Euclidean plane containing u but not containing a , as a symmetric bilinear form. The λ -function is a determinant function with u and a as two of its arguments, except in the last case [6.17] where $\lambda_x = -s(u, x)$ for all x .

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1. For definition of non-definite forms, see Bocher's Introduction to Higher Algebras, p. 162.
 2. For definition and properties of Pseudo-Euclidean space, see Tullio Levi-Civita's A Simplified Presentation of Einstein's Unified Field Equations, p. 26.
 3. For definitions of positive definite forms, see Bocher's Introduction to Higher Algebras, p. 162.

§ 7 ONE WAY COMMUTATIVE ALGEBRAS IN 4-DIMENSIONAL SPACE

Let us choose a_1, a_2 and a_3 from the space so that they together with u introduced in §6 are four linearly independent vectors. Then all $\phi(x, y)$, x, y ranging through the entire space, can be expressed as follows: $\phi(x, y) = c^1_{xy}a_1 + c^2_{xy}a_2 + c^3_{xy}a_3 + c^4_{xy}u$ where the c^i_{xy} 's are skew bilinear scalar function of x, y . By Theorem 9, $c^i_{xy} = [a^i b^i]_{xy} + [c^i d^i]_{xy}$. But $c^i_{uy} = 0$ for all y , hence if a^i, b^i and u (or c^i, d^i and u) are linearly independent, $c^i_{uc^i} = [a^i b^i]_{uc^i} = 0$, $c^i_{ud^i} = [a^i c^i]_{ud^i} = 0$, i.e. c^i and d^i lie in the same space with a^i, b^i and u .

$$\begin{aligned} \therefore [c^i, d^i]_{x, y} &= [\xi_1 a^i + \xi_2 b^i + \xi_3 u, \eta_1 a^i + \eta_2 b^i + \eta_3 u]_{x, y} \\ &= (\xi_1 \eta_2 - \xi_2 \eta_1) [a^i b^i]_{xy} + [u, \xi_3 (\eta_1 a^i + \eta_2 b^i) - \eta_3 (\xi_1 a^i + \xi_2 b^i)]_{x, y} \\ \therefore c^i_{xy} &= [a^i, b^i]_{x, y} + [u, c^i]_{x, y} \end{aligned}$$

where $b^i = b^i + (\xi_1 \eta_2 - \xi_2 \eta_1) b^i$, $c^i = \xi_3 (\eta_1 a^i + \eta_2 b^i) - \eta_3 (\xi_1 a^i + \xi_2 b^i)$. Then $c^i_{uy} = [a^i, b^i]_{u, y} = 0$ for all y , hence a^i, b^i, u must be coplanar, or $a^i = \alpha_1 d^i + \alpha_2 u$ and $b^i = \beta_1 a^i + \beta_2 u$, d^i being a vector in that plane not collinear with u .

Consequently, $c^i_{xy} = [u, \alpha_2 \beta_1 d^i - \beta_2 \alpha_1 d^i]_{x, y}$

$$+ [u, c^i]_{x, y} = [u, a^i]_{x, y} \text{ where } a^i = \alpha_2 \beta_1 d^i - \beta_2 \alpha_1 d^i + c^i$$

If neither a^i, b^i and u , nor c^i, d^i and u are linearly independent, then it is obvious that c^i_{xy} reduces right away to the form we just secured for it. Therefore we can always write

$$\phi(x, y) = [b_1 u x y] a_1 + [b_2 u x y] a_2 + [b_3 u x y] a_3 + [b_4 u x y] u.$$

Let $b_1 = B_{11}a_1 + B_{12}a_2 + B_{13}a_3 + B_{14}u$, $b_2 = B_{21}a_1 + B_{22}a_2 + B_{23}a_3 + B_{24}u$, etc.

Then $\phi(x, y) = (B_{11}[a_1 u x y] + B_{12}[a_2 u x y] + B_{13}[a_3 u x y])a_1$

$$\begin{aligned}
& + (B_{21}[a_1 u x y] + B_{22}[a_2 u x y] + B_{23}[a_3 u x y]) a_2 \\
& + (B_{31}[a_1 u x y] + B_{32}[a_2 u x y] + B_{33}[a_3 u x y]) a_3 \\
& + (B_{41}[a_1 u x y] + B_{42}[a_2 u x y] + B_{43}[a_3 u x y]) u \\
& = [a_1 u x y] (B_{11} a_1 + B_{21} a_2 + B_{31} a_3 + B_{41} u) \\
& + [a_2 u x y] (B_{12} a_1 + B_{22} a_2 + B_{32} a_3 + B_{42} u) \\
& + [a_3 u x y] (B_{13} a_1 + B_{23} a_2 + B_{33} a_3 + B_{43} u)
\end{aligned}$$

$$\text{or } [7.1] \phi(x, y) = [a_1 u x y] a_1' + [a_2 u x y] a_2' + [a_3 u x y] a_3'$$

where, as we mentioned in the beginning, a_1, a_2 and a_3 together with u are four linearly independent vectors. We can show that a_1', a_2' , and a_3' must also be linearly independent of each other. For, were they not so, they would form a plane either containing u or linearly independent of u ; in the first case, we would have $a_1 = \alpha_{11}u + \alpha_{12}v$, $a_2' = \alpha_{21}u + \alpha_{22}v$, $a_3' = \alpha_{31}u + \alpha_{32}v$, and $\phi(x, y) = [\alpha_{11}a_1 + \alpha_{21}a_2 + \alpha_{31}a_3, u, x, y]u + [\alpha_{12}a_1 + \alpha_{22}a_2 + \alpha_{32}a_3, u, x, y]v$ for all x and y , violating Theorem 13; in the second case, $a_3' = \gamma_{1a_1}' + \gamma_{2a_2}'$ where a_1', a_2' , and u are linearly independent (unless a_1' , and a_2' are collinear, but then $\phi(x, y)$ would all lie in one direction, violating Theorem 5), and hence $\phi(a_1', a_2') \neq 0$, but $\phi(x, y) = [a_1 + \gamma_{1a_3, u, x, y}, a_1' + [a_2 + \gamma_{2a_3, u, x, y}] a_2'] = [b_1' u, x, y] a_1' + [b_2' u, x, y] a_2'$ and the Jacobian Identity gives $0 = [b_1' u x y] ([b_1' u a_1' z] a_1' + [b_2' u a_1' z] a_2') + [b_2' u x y] ([b_1' u a_2' z] a_1' + [b_2' u a_2' z] a_2') + [b_1' u z x] ([b_1' u a_1' y] a_1' + [b_2' u a_1' y] a_2') + [b_2' u z x] ([b_1' u a_2' y] a_1' + [b_2' u a_2' y] a_2') + [b_1' u y z] ([b_1' u a_1' x] a_1' + [b_2' u a_1' x] a_2') + [b_2' u y z] ([b_1' u a_2' x] a_1' + [b_2' u a_2' x] a_2') = [b_2', u, a_1', [x y z u] b_1'] a_2' + [b_1', u, a_2', [x y z u] b_2'] a_1'$

making use of Theorem 2,

$\therefore [x y z u] [b_2' u a_1' b_1'] = 0$ and $[x y z u] [b_1' u a_2' b_2'] = 0$, and since x, y, z are arbitrary, $\therefore [b_2' u a_1' b_1'] = [b_1' u a_2' b_2'] = 0$. consequently $[b_1' u a_1' a_2'] = 0$ and $[b_2' u a_1' a_2'] = 0 \therefore \phi(a_1', a_2') = 0$, a contradiction. Hence we formulate the following theorem.

Theorem 16. In one way commutative algebras in 4-di-

mensional space, the skew part of the associative operation σ has the form $\phi(x,y)=[a_1uxy]a_1'+[a_2uxy]a_2'+[a_3uxy]a_3'$ for any x,y in the space, where the $a_i',s, i=1,2,3$, are linearly independent and a_1,a_2,a_3 and u are linearly independent.

1. Suppose u is linearly independent of a_1',a_2' , and a_3' .

If $\lambda u=O$, then $\lambda x=O$ for all x since we can always express x in terms of u,a_1',a_2' and a_3' and by [7.1] $\phi(a_i,a_j)=[a_kua_ia_j]a'_k, i=1,2,3, j=1,2,3, k=1,2,3, i\neq j\neq k\neq i$, and hence $\lambda a'_i=O$ for $i=1,2,3$ by [6.2]. Then [6.1] becomes $\theta(x,y)=s(x,y)u$ for any x and y . Therefore [6.3] becomes $\lambda u=\phi(\phi(z,x),y)$, λ being some scalar. But for some z and x , $\phi(\phi(z,x),y)$ has its zero space of dimension 2 and hence its value space of dimension 2 also, by Theorem 5.

So we have a contradiction. Therefore $\lambda u\neq O$. [6.4] requires then $\lambda z+s(z,u)=O$ for all z , or $\lambda z=-s(u,z)$ for any z . And [6.1] becomes [7.2] $\theta(x,y)=s(x,y)u-s(y,u)x-s(x,u)y$,

while [6.3] becomes [7.3]
$$\begin{vmatrix} s(z,u)s(x,u)s(u,u) \\ s(z,y)s(x,y)s(u,y) \\ z \quad x \quad u \end{vmatrix} = \phi(\phi(z,x),y)$$

If we replace a_1',a_2',a_3' by any other set of linearly independent vectors in the subspace formed by linear combinations of a_1',a_2' , and a_3' with a_1,a_2,a_3 replaced correspondingly by certain linear combinations of these vectors, the form of [7.1] will be the same, all relations we obtained so far are not affected, and still u will be linearly independent of the new set of a'_i 's, and the new a'_i 's and u will still be linearly independent. In that subspace we just mentioned there must be some vector, say A_1 , such that $s(A_1,A_1)\neq O$; for otherwise $s(x,y)$ would vanish for all x and y in that subspace, and, sa by [6.2] $s(a'_i,u)=O$ for $i=1,2,3$, the left hand side of [7.3] would vanish for all x,y,z of that subspace and yet in the right hand

side $\phi(z, x)$ gives a non-zero vector in that subspace for any noncollinear z and x in the same and hence $\phi(\phi(z, x), y)$ will not vanish for some y in the same, or we have a contradiction. As a linear function of x , $s(A_1, x)$ has a plane in that subspace on which the function vanishes. For the same reason, we should have at least one vector in that plane, say A_2 , such that $s(A_2, A_2) \neq 0$, whereas $s(A_1, A_2) = 0$. Likewise, as a linear function of x , $s(A_2, x)$ has a plane in the same subspace, on which the function vanishes. Now, as in the subspace of dimension 3, this plane must have one direction in common with the first plane. Take a vector A_3 in that direction, then $s(A_1, A_3) = s(A_2, A_3) = 0$, and A_1, A_2, A_3 have to be linearly independent. We can show that $s(A_3, A_3) \neq 0$: Assume the contrary. Then for any vector x of the same subspace $s(A_3, x) = 0$. Now either $\phi(A_3, A_1)$ or $\phi(A_3, A_2)$ must not be collinear with A_3 ; for otherwise would $\phi(A_3, A_1) = \lambda \phi(A_3, A_2)$ or $\phi(A_3, A_1 - \lambda A_2) = 0$; since A_1, A_2, A_3 and u are linearly independent, this is impossible. Suppose $\phi(A_3, A_i)$, $i=1$ or 2 , is not collinear with A_3 . Then by [7.1] $\phi(A_3, A_i)$ gives a non-zero vector in the same subspace linearly independent of A_3 and u . Hence $\phi(\phi(A_3, A_i), A_3) \neq 0$. But if we set $z=y=A_3, x=A_i$ in [7.3], we get $0 = \phi(\phi(A_3, A_i), A_3)$. Hence a contradiction; or $s(A_3, A_3) \neq 0$.

So if we replace a_1', a_2', a_3' by A_1, A_2, A_3 , we have

$$[7.4] \quad \phi(x, y) = [A_1' uxy] A_1 + [A_2' uxy] A_2 + [A_3' uxy] A_3$$

where $s(A_i, A_j) = 0$ when $i \neq j$ and $\neq 0$ when $i = j$; and by [6.2], $s(u, \phi(A_j', A_k')) = 0$, or $s(u, A_i) = 0$, $i, j, k = 1, 2, 3$; $i \neq j \neq k \neq i$. Also $s(u, u) \neq 0$, since $\lambda_u \neq 0$ and by the relation [6.4]. Hence no x in the space can make $s(x, y)$ vanish for all y ; for, if there were $x = x^1 A_1 + x^2 A_2 + x^3 A_3 + x^4 u$ such that $s(x, y) = 0$ for all y , then $s(x, A_1) = s(x, A_2) = s(x, A_3) = s(x, u) = 0$, or $x^1 = x^2 = x^3 = x^4 = 0$, i. e. $x = 0$. Then by Theorem 6, there exists for z, x, u a skew trilinear vector function $[zxu]$ such

that $s([zxu], z) = s([zxu], x) = s([zxu], u) = 0$. We see that $\phi(z, x)$ has also these properties and since $[zxu]$ and $\phi(z, x)$ are both linearly independent of z , x , and u , they must be the same direction, otherwise we would have $[zxu] = \alpha z + \beta x + \gamma u + \rho \phi(z, x)$ and $([zxu] - \rho \phi(z, x))$ would be a non-zero vector denoted by v such that $s(v, y) = 0$ for all y which is impossible as shown before. By comparing [7.3] with the general form in Theorem 6, we see that $\phi(z, x)$ is nothing but the triple vector product¹ of u, x , and z for any z and x in the space. And we can use $[zxu]$ to denote $\phi(z, x)$ for any z, x .

If we set $z = A_1, x = A_2, y = A_3$ in [7.3], the left side of the equation vanishes as $s(A_1, A_3) = s(A_2, A_3) = s(u, A_3) = 0$.

So we have

$$0 = \phi(\phi(A_1, A_2), A_3) = [A_1' u A_1 A_2] ([A_1' u A_1 A_3] A_1 + [A_2' u A_1 A_3] A_2 + [A_3' u A_1 A_3] A_3) + [A_2' u A_1 A_2] ([A_1' u A_2 A_3] A_1 + [A_2' u A_2 A_3] A_2 + [A_3' u A_2 A_3] A_3)$$

by [7.4]. Since A_1, A_2 , and A_3 are linearly independent, their coefficients must vanish. Considering the coefficient of A_1 , we have

$$[A_1' u A_1 A_2] [A_1' u A_1 A_3] + [A_2' u A_1 A_2] [A_1' u A_2 A_3] = 0$$

Using Theorem 2, we see that

$$\begin{aligned} [A_2' u A_1 A_2] [A_1' u A_2 A_3] &= [A_1', u, A_2, [A_2' u A_1 A_2] A_3] \\ &= [A_1', u_1, A_2, [A_3' u A_1 A_2] A_2' + [A_2' u A_3 A_2] A_1] \\ &= [A_1' u A_2 A_2'] [A_3' u A_1 A_2] + [A_1' u A_2 A_1] [A_2' u A_3 A_2] \end{aligned}$$

Therefore we have

$$\begin{aligned} &[A_1' u A_1 A_2] [A_1' u A_1 A_3] + [A_1' u A_2 A_2'] [A_3' u A_1 A_2] \\ &+ [A_1' u A_2 A_1] [A_2' u A_3 A_2] = 0 \\ \text{or [7.5]} &[A_1' u A_1 A_2] ([A_1' u A_1 A_3] + [A_2' u A_2 A_3]) \\ &+ [A_1' u A_2 A_2'] [A_3' u A_1 A_2] = 0 \end{aligned}$$

By setting $x = y = A_3, z = A_1$, in [7.3], we have

$$\begin{aligned} [7.6] - s(u, u) s(A_3, A_3) A_1 &= \phi(\phi(A_1, A_3), A_3) \\ &= [A_1' u A_1 A_3] ([A_1' u A_1 A_3] A_1 + [A_2' u A_1 A_3] A_2 + [A_3' u A_1 A_3] A_3) \end{aligned}$$

+ [A₂'uA₁A₃]([A₁'uA₂A₃]A₁ + [A₂'uA₂A₃]A₂ + [A₃'uA₂A₃]A₃)
 Since $s(u, u) \neq 0$, $s(A_3 A_3) \neq 0$, hence in the right hand side the coefficient of A₁ must not vanish, while the coefficients of A₂ and A₃ must vanish. Suppose [A₂'uA₁A₃] = 0. Then [A₁'uA₁A₃] must not vanish, but [A₂'uA₁A₃] must vanish. Yet by setting x=y=A₁, z=A₃ in [7.3], we have

$$\begin{aligned} -s(u, u)s(A_1, A_1)A_3 &= \phi(\phi(A_3, A_1), A_1) \\ &= [A_2'uA_3A_1]([A_1'uA_2A_1]A_1 + [A_2'uA_2A_1]A_2 + [A_3'uA_2A_1]A_3) \\ &\quad + [A_3'uA_3A_1]([A_1'uA_3A_1]A_1 + [A_2'uA_3A_1]A_2 + [A_3'uA_3A_1]A_3) \end{aligned}$$

Here, since $s(A_1, A_1) \neq 0$ and

$$[A_2'uA_3A_1] = -[A_2'uA_1A_3] = 0, \quad [A_3'uA_3A_1]$$

must not vanish. As $[A_3'uA_3A_1] = -[A_3'uA_1A_3] = 0$, we have a contradiction. Therefore $[A_2'uA_1A_3] \neq 0$. Similarly we can show that $[A_3'uA_1A_2] \neq 0$. Since the coefficient of A₂ in [7.6] must vanish,

$$\begin{aligned} \therefore [A_2'uA_1A_2]([A_1'uA_1A_3] + [A_2'uA_2A_3]) &= 0, \quad \text{or} \\ [A_1'uA_1A_3] + [A_2'uA_2A_3] &= 0 \quad \text{as } [A_2'uA_1A_3] \neq 0. \end{aligned}$$

Hence [7.5] becomes $[A_1'uA_2A_2'] [A_3'uA_1A_2] = 0$, or $[A_1'uA_2A_2'] = 0$ as $[A_3'uA_1A_2] \neq 0$. Similarly we can show that $[A_3'uA_2A_2'] = 0$. Since also $[A_2'uA_2A_2'] = 0$ and $[u uA_2A_2'] = 0$ where A₁', A₂', A₃', and u are linearly independent, $\therefore [xuA_2A_2'] = 0$ for all x in the space. Therefore, by Theorem 3, u, A₂ and A₂' are linearly dependent, or $A_2' = \mu A_2 + \mu' u$. Similarly we can show that $A_1' = \lambda A_1 + \lambda' u$, and $A_3' = \nu A_3 + \nu' u$, where $\lambda, \lambda', \mu, \mu', \nu, \nu'$ are all scalars. Then [7.4] becomes

$$\phi(x, y) = \lambda[A_1 uxy]A_1 + \mu[A_2 uxy]A_2 + \nu[A_3 uxy]A_3,$$

or by setting

$$B_1 = \sqrt{\lambda} A_1, \quad B_2 = \sqrt{\mu} A_2, \quad B_3 = \sqrt{\nu} A_3, \quad \text{we have}$$

$$[7.7] \quad \phi(x, y) = [B_1 uxy]B_1 + [B_2 uxy]B_2 + [B_3 uxy]B_3$$

Thus we can formulate the following theorem:

Theorem 17. In a 1-way commutative algebra in 4-dimen-

sional space, if the ϕ -function ranges throughout a 3-dimensional subspace not containing u , then it has the form [7.7] where B_1, B_2, B_3 and u are four linearly independent vectors and $s(B_i, B_j) = 0$ for $i \neq j$, $s'(B_i, B_i) \neq 0$, and $s(B_i, u) = 0$, $i = 1, 2, 3$, $j = 1, 2, 3$.

Now [7.6] yields to

$$\begin{aligned} -s'(u, u)s(B_3, B_3)B_1 &= [B_2uB_1B_3][B_1uB_2B_3]B_1 = -[uB_1B_2B_3]^2B_1 \\ \therefore s(u, u)s(B_3, B_3) &= [uB_1B_2B_3]^2; \text{ likewise,} \\ s(u, u)s(B_1, B_1) &= [uB_1B_2B_3]^2, \text{ and } s(u, u)s(B_2, B_2) \\ &= [uB_1B_2B_3]^2. \end{aligned}$$

Therefore in this space the s -function is positive definite, and we can fix the lengths of u, B_1, B_2 , and B_3 so that $s(u, u) = s(B_1, B_1) = s(B_2, B_2) = s(B_3, B_3) = 1$. If we write for any x, y , $x = x^1B_1 + x^2B_2 + x^3B_3 + x^4u$ and $y = y^1B_1 + y^2B_2 + y^3B_3 + y^4u$ we have

$$s(x, y) = x^1y^1 + x^2y^2 + x^3y^3 + x^4y^4,$$

and [7.2] becomes

$$\begin{aligned} \theta(x, y) &= (x^1y^1 + x^2y^2 + x^3y^3 + x^4y^4)u - y^4x - x^4y \\ &= (x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4)u - (x^1y^4 + x^4y^1)B_1 \\ &\quad - (x^2y^4 + x^4y^2)B_2 - (x^3y^4 + x^4y^3)B_3 \end{aligned}$$

[7.7] becomes

$$\begin{aligned} \phi(x, y) &= (x^3y^2 - x^2y^3)[uB_1B_2B_3]B_1 + (x^1y^3 - x^3y^1)[uB_1B_2B_3]B_2 \\ &\quad + (x^2y^1 - x^1y^2)[uB_1B_2B_3]B_3 \\ &= (x^3y^2 - x^2y^3)B_1 + (x^1y^3 - x^3y^1)B_2 + (x^2y^1 - x^1y^2)B_3 \end{aligned}$$

if we take $[uB_1, B_2, B_3] = 1$.

In fixing the lengths of B_i 's, $i = 1, 2, 3$, so that $s(B_i, B_i) = 1$, this form of ϕ may bear a scalar multiplier which is not essential. Hence

$$\begin{aligned} [7.8] \quad \sigma(x, y) &= (x^1y^1 + x^2y^2 + x^3y^2 - x^4y^4)u - (x^1y^4 + x^4y^1 + x^2y^3 \\ &\quad - x^3y^2)B_1 - (x^2y^4 + x^4y^2 + x^3y^1 - x^1y^3)B_2 \\ &\quad - (x^3y^4 + x^4y^3 + x^1y^2 - x^2y^1)B_3 \end{aligned}$$

We observe that $\sigma(u, u) = -u$, $\sigma(B_i, B_i) = u$,

$$\sigma(u, B_i) = -B_i, \sigma(B_i, B_j) = -B_k$$

where $i=1,2,3$, $j=1,2,3$, $k=1,2,3$, $i \neq j \neq k \neq i$. Or this is an algebra of quaternions where $u=-1$, $B_1=i$, $B_2=j$, $B_3=k$, $1, i, j$, and k being ordinary symbols of its units.

2. Suppose u , and a_1' , a_2' , and a_3' as appear in [7.1] are not linearly independent. Then $\phi(x,y)$ for all x and y must have the form $\phi(x,y)=[A_1uxy]A_1'+[A_2uxy]A_2'+[A_3uxy]u$ where A_1', A_2' , and u are linearly independent and A_1, A_2, A_3 and u are linearly independent, otherwise the case would reduce to those considered before. Then [6.2] gives $\lambda_{A_1}=\lambda_{A_2}=\lambda_u=0$, and $s(u,u)=s(u,A_1')=s(u,A_2')=0$.

Since we see that if we replace A_1', A_2' by any other two vectors in the subspace formed by linear combinations of A_1', A_2' , and u such that the new ones and u still form a linearly independent set, all relations true for A_1' and A_2' will be also true for the new ones, we are going to make such changes several times.

Suppose $s(A_1', A_2') \neq 0$. Then we shall replace A_1' and A_2' by b_1', b_2' , in the following manner: If $s(A_1', A_1') = s(A_2', A_2') = 0$, then we set $b_1' = A_1' + A_2'$, and $b_2' = A_1' - A_2'$; if either $s(A_1', A_1')$ or $s(A_2', A_2')$ differs from zero, say $s(A_1', A_1') \neq 0$ (argument same for $s(A_2', A_2') \neq 0$), then we set $b_1' = A_1'$, and $b_2' = A_2' - \frac{s(A_1', A_2')}{s(A_1', A_1')} A_1'$. In either case we have $s(b_1', b_2') = 0$.

Suppose $s(A_1', A_2') = 0$. Then we shall rename A_1', A_2' by b_1', b_2' respectively. Hence anyway we have [7.9] $\phi(x,y) = [b_1uxy]b_1' + [b_2uxy]b_2' + [b_3uxy]u$ where b_1 and b_2 are the corresponding substitutes for A_1 and A_2 , b_3 is simply A_3 , and $s(b_1', b_2') = s(u, b_1') = s(u, b_2') = s(u, u) = 0$.

Let b_4' be any vector of the space such that b_1', b_2', b_4' and u are linearly independent. Setting $z=b_2'$, $y=b_1'$, $x=b_4'$ in [6.3], we have

$$[7.10] \quad O = [b_2 u b'_2 b'_4]([b_1 u b'_2 b'_1] b'_1 + [b_2 u b'_2 b'_1] b'_2 + [b_3 u b'_2 b'_1] u).$$

The Jacobian Identity gives

$$[b_{\alpha} u x y][b_{\beta} u b'_{\alpha} z] b'_{\beta} + [b_{\alpha} u y z][b_{\beta} u b'_{\alpha} x] b'_{\beta} + [b_{\alpha} u z x][b_{\beta} u b'_{\alpha} y] b'_{\beta} = O$$

where for convenience of notation we let $b'_3 = u$ temporarily, and α and β denote the summation from 1 to 3.

Then $[b_1, u, b'_{\alpha}, [b_{\alpha} u x y] z + [b_{\alpha} u y z] x + [b_{\alpha} u z x] y] = O$

or, using Theorem 2, $[b_1, u, b'_{\alpha}, b_{\alpha}][u x y z] = O$;

since x, y, z are arbitrary, $\therefore [b_1, u, b'_{\alpha}, b_{\alpha}] = O, i = 1, 2, 3$.

Remembering that $b'_3 = u$, we have

$$[b_1 u b'_2 b_2] = O, [b_2 u b'_1 b_1] = O, \text{ and } [b_3 u b'_1 b_1] + [b_3 u b'_2 b_2] = O.$$

From the first two equations it follows that

$$[b_1 u b'_2 b'_1] = [b_2 u b'_2 b'_1] = O. \text{ Hence [7.10] becomes}$$

$$O = [b_2 u b'_2 b'_4][b_3 u b'_2 b'_1] u. \text{ Now } [b_3 u b'_2 b'_1] \neq O, \text{ for [7.9] gives}$$

$\phi(b'_2, b'_1) = [b_3 u b'_2 b'_1] u$ which can not vanish since b'_1 and b'_2 are linearly independent of u and not collinear themselves.

Therefore $[b_2 u b'_2 b'_4] = O$. Likewise, $[b_1 u b'_1 b'_4] = O$. Since

also $[b_1 u b'_1 b'_1] = O$, $[b_1 u b'_1 b'_2] = O$, and $[b_1 u b'_1 u] = O$, hence $[b_1 u b'_1 x] = O$ for all x . By Theorem 3, $\therefore b_1 = \lambda b'_1 + \lambda' u$, and

similarly $\mu_2 = \mu b'_2 + b' u$. Hence if we take $B_1 = \sqrt{\lambda} b'_1$

$B_2 = \sqrt{\mu} b'_2$, $B_3 = b_3$, [7.9] becomes

$$[7.11] \quad \phi(x, y) = [B_1 u x y] B_1 + [B_2 u x y] B_2 + [B_3 u x y] u.$$

Theorem 18. In a 1-way commutative algebra of 4-dimensional space, if ϕ ranges throughout a 3-dimensional supspace containing u , then it has the form [7.11] where $s(B_1, B_2)$

$$= s(u, B_1) = s(u, B_2) = s(u, u) = O.$$

Setting $x = y = B_1$, $z = B_3$ in [6.3], we get

$$s(B_1, B_1) s(u, B_3) = [B_2 u B_3 B_1][B_3 u B_2 B_1] = -[u B_1 B_2 B_3]^2; \text{ and}$$

likewise,

$$[7.12] s(B_2, B_2) s(u, B_3) = [B_1 u B_3 B_2] [B_3 B_1 B_2] = -[u B_1 B_2 B_3]^2.$$

Since u, B_1, B_2 , and B are linearly independent,

$$[u B_1 B_2 B_3] \neq 0, \quad s(B_1, B_1) = s(B_2, B_2) \neq 0, \quad \text{and} \quad s(u, B_3) \neq 0.$$

Setting $x = B_1, y = z = B_3$, in [6.3], we have

$$-\lambda_{B_3} \lambda_{B_3} B_1 + s(B_1, B_3) s(u, B_3) u = [B_2 u B_3 B_1] [B_1 u B_2 B_3] B_1$$

$$\therefore [7.13] -\lambda_{B_3}^2 = [u B_1 B_2 B_3]^2, \quad \text{or} \quad \lambda_{B_3} = \pm i [u B_1 B_2 B_3] \quad \text{where} \\ i = \sqrt{-1}; \quad \text{and} \quad [7.14] \quad s(B_1, B_3) = 0, \quad \text{and likewise} \quad s(B_2, B_3) = 0.$$

Thus we see that for any $x \neq 0$ in this space $s(x, y)$ will not vanish for all y . By Theorem 6, there exists a trilinear skew vector function $[x, y, z]$ such that $s([xyz], w)$ is a determinant function. So $s([zxu], z) = s([zxu], x) = s([zxu], u) = 0$. We can easily show as we did before that $[zxu]$ is in the direction of $\phi(z, x)$. Hence we shall write $\phi(z, x) = [zxu]$ for any z, x . Then [6.1] becomes [7.15] $\theta(x, y) = \pm i [u B_1 B_2 y]_x \pm i [u B_1 B_2 x]_y + s(x, y)u$ where $s(x, y)$ is a non-definite symmetric bilinear form, and [7.16] $\sigma(x, y) = \pm i [u B_1 B_2 y]_x \pm i [u B_1 B_2 x]_y + s(x, y)u + [xyu]$

From [7.12] we see that this space is Pseudo-Euclidean. It is possible to choose from the plane containing u and B_3 two vectors B'_3 , and B_4 such that $s(B'_3, B_4) = 0$ and $s(B'_3, B'_3) = -s(B_4, B_4) \neq 0$.

Still B_1, B_2, B'_3 , and B_4 will be linearly independent, and $s(B_i, B'_3) = s(B_i, B_4) = 0, i = 1, 2$. If we fix their lengths so that $s(B_1, B_1) = s(B_2, B_2) = s(B'_3, B'_3) = 1$, and $s(B_4, B_4) = -1$, making the expression [7.11] for $\phi(x, y)$ bear a scalar multiplier, we can have for any $x = x^1 B_1 + x^2 B_2 + x^3 B'_3 + x^4 B_4$ and $y = y^1 B_1 + y^2 B_2 + y^3 B'_3 + y^4 B_4$, [7.17] $s(x, y) = x^1 y^1 + x^2 y^2 + x^3 y^3 - x^4 y^4$.

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1. The Triple product is a special case of the combinatory product as defined and discussed by H. Grassmann; See J. V. Collins's *An Elementary Exposition of Grassmann's Ausdehnungslehre*, pp. 11-15.

§ 8. 2-WAY COMMUTATIVE ALGEBRAS

Here the space has a centrum of dimension 2. Hence $K_{x,y}$ of [5.1] becomes

$$s_1(x,y)u_1 + s_2(x,y)u_2$$

where u_1 and u_2 are any two linearly independent vectors in the centrum and $s_1(x,y)$ and $s_2(x,y)$ are symmetric bilinear scalar functions. Hence [5.1] becomes

$$[8.1] \quad \theta(x,y) = \lambda_{yx} + \lambda_{xy} + s_1(x,y)u_1 + s_2(x,y)u_2;$$

and [5.2] gives

$$[8.2] \quad \lambda_{\phi(x,y)} = 0, \quad s_i(\phi(x,y),z) = s_i(\phi(y,z),x) = s_i(\phi(z,x),y), \\ s_i(\phi(x,y),u_i) = s_i(\phi(x,y),x) = 0, \text{ for any}$$

$$x,y,z, \text{ where } i,j = 1, 2.$$

Also [5.3] becomes

$$[8.3] \quad (\lambda_{xy} + s_1(x,y)\lambda_{u_1} + s_2(x,y)\lambda_{u_2})z - (\lambda_{zy} + s_1(x,y)\lambda_{u_1} \\ + s_2(z,y)\lambda_{u_2})x \\ + \{s_1(x,y)s_1(u_1,z) + s_2(x,y)s_1(u_2,z) - s_1(z,y)s_1(u_1,x) \\ - s_2(z,y)s_1(u_2,x)\}u_1 \\ + \{s_1(x,y)s_2(u_1,z) + s_2(x,y)s_2(u_2,z) - s_1(z,y)s_2(u_1,x) \\ - s_2(z,y)s_2(u_2,x)\}u_2 = \phi(\phi(z,x),y)$$

In a space of dimension 2, the 2-way commutative algebras are also 1-way commutative, hence they have been discussed in the previous section §6. So there are only four cases to be considered, i.e. the dimension of the space $n=3$, $n=4$, $n=5$, and $n=6$, since by Theorem 12 there are no associative 2-way-commutative algebras in spaces of dimension $n \geq 7$. The case $n=3$ gives an entirely commutative algebra, provided the

latter exists, where the ϕ -function is zero identically and the σ -function will be symmetric. In the case $n=4$, the ϕ -function should lie in a fixed direction and we can easily derive by a procedure similar to that used in §6 $\phi(x,y)=[u_1u_2xy]$ where $[xyzw]$ denotes a determinant function. While no further investigation is made in this work, it is hoped that there exist associative algebras in each of the four cases.

For three or more way commutative algebras, the investigation is also left open.

§ 9 ALGEBRAS IN DIFFUSED SPACES

As Albert defines abelian groups and subgroups,¹ we shall define abelian subspaces in the similar way, thus: an *abelian subspace* is a subspace in which all vectors are commutative with each other; a *complete abelian subspace* is an abelian subspace outside which no vector is commutative with all the vectors in it. Spaces containing algebras that are not entirely commutative, nor entirely non-commutative, nor centralized, will be called *diffused spaces*. In these spaces the skew part of the associative operation σ , i.e. the ϕ -function though not vanishing identically, may vanish in a great variety of ways. There may be abelian subspaces not all of which are commutative with same set of elements in the algebra. We shall define a *complete commutative subspace of x* as a subspace which contains all vectors of the space that are commutative with x except x itself.

Using again the equation $\phi(\theta(x,x),x)=0$ obtained by setting $y=z=x$ in [3.5], we can still get a general expression for $\theta(x,x)$, i.e. [9.1] $\theta(x,x)=2\lambda_{xx}+K^r_{xx}$ where λ_x is still a linear scalar function of x and K^r_{xx} is a vector in the complete commutative subspace of x which has dimension r .

Theorem 19. If the complete commutative subspace of y is of dimension s and contains the r -dimensional complete commutative subspace of x , but does not contain x , i.e. x is not commutative with y , or $\phi(x,y)\neq 0$ then $\theta(y,y)=2\lambda_{yy}=K'^r_{yy}$ where $\theta(x,x)=2\lambda_{xx}+K^r_{xx}$; and $\theta(x,y)=\lambda_yx+\lambda_{xy}-K''^r_{yx}$; K^r_{xx} ,

$K'^{r_{xx}}$, and $K''^{r_{xx}}$ are all vectors in the complete commutative subspace of x .

Proof. Setting $z=x$ in [3.4], we have

$$\phi(\theta(x,y),y) + 2\phi(\theta(x,y),x) = 0,$$

$$\text{or } \phi(2\lambda_{xx}x + K'^{r_{xx}},y) + 2\phi(\theta(x,y),x) = 0,$$

$$\text{or } 2\lambda_{xx}\phi(x,y) + 2\phi(\theta(x,y),x) = 0.$$

or $\phi(x, \lambda_{xy}y - \theta(x,y)) = 0$; $\therefore \lambda_{xy}y - \theta(x,y) = l x + K''^{r_{xx}}$ where l is some scalar and $K''^{r_{xx}}$ is a vector in the complete commutative subspace of x , so that $\phi(x, K''^{r_{xx}}) = \phi(y, K''^{r_{xx}}) = 0$.

Hence $\theta(x,y) = \lambda_{xy}y - l x - K''^{r_{xx}}$.

Now setting $z=y$ in [3.4], we have

$$\phi(\theta(y,y),x) + 2\phi(\theta(x,y),y) = 0,$$

$$\text{or } \phi(\theta(y,y),x) - 2l\phi(x,y) = 0$$

or $\phi(\theta(y,y) + 2l y, x) = 0$, $\theta(y,y) + 2l y = m x + K'''^{r_{xx}}$ where m is some scalar and $K'''^{r_{xx}}$ is a vector lying in the complete commutative subspace of x . Hence $\theta(y,y) = m x + K'''^{r_{xx}} - 2l y$.

But by hypothesis, $\theta(y,y) = 2\lambda_{yy}y + K'^{r_{yy}} = 2\lambda_{yy}y + (K^{s-r_{yy}} + K'^{r_{xx}})$ where $K'^{r_{xx}}$ is a vector in the complete commutative subspace of x which does not contain $K^{s-r_{yy}}$. Therefore $m=0$, $l = -\lambda_y$, and $K'''^{r_{xx}} = K'^{r_{xx}}$, or

$$[9.2] \quad \theta(y,y) = 2\lambda_{yy}y + K'^{r_{xx}}; \text{ and}$$

$$[9.3] \quad \theta(x,y) = \lambda_{xy}y + \lambda_{yx}x - K''^{r_{xx}}.$$

Theorem 20. If in any diffused space there exists one x such that no y except in the direction of x makes $\phi(x,y)$ vanish, or, in other words, x is commutative only with vectors in its own direction, then the general form of $\theta(y,z)$ for all y and z in the space is:

$$[9.4] \quad \theta(y,z) = \lambda_y y + \lambda_z z.$$

Proof. By Theorem 19, where $K'^{r_{xx}}$ has dimension $r=0$, we have $\theta(y,y) = 2\lambda_{yy}y$ for all y in the space.

$$\text{Hence } \theta(y+z, y+z) = 2\lambda_{y+z}(y+z)$$

$$\text{But } \theta(y+z, y+z) = \theta(y,y) + \theta(z,z) + 2\theta(y,z)$$

$$= 2\lambda_{yy} + 2\lambda_{zz} + 2\theta(y, z),$$

$$\theta(y, z) = \lambda_{y+z}(y+z) - \lambda_{yy} - \lambda_{zz} = \lambda_{zy} + \lambda_{yz}$$

Since, being linear, $\lambda_{y+z} = \lambda_y + \lambda_z$, for all y and z .

Theorem 21. If an associative algebra has one complete abelian subspace of dimension 1, then it must have also an abelian subspace of $n-1$ dimensions, where n is the dimension of the space considered.

Proof. Under the restriction on the algebra as mentioned in the theorem, the space can be either entirely non-commutative or diffused. In either case, as we have seen, in §4 and in Theorem 20, the general form of $\theta(x, y)$ for all x and y in the space is: $\theta(x, y) = \lambda_y x + \lambda_{xy}$. By Theorem 4, the λ -function has an $(n-1)$ -dimensional subspace on which it vanishes. Suppose the algebra has no abelian subspace of $n-1$ dimensions. Let us denote by x that vector which is non-commutative with every other vector in different direction. Then there will be at least two vectors, say x' and z , in the $(n-1)$ -dimensional subspace, which are not commutative with each other, where $\lambda_{x'} = 0$ and $\lambda_z = 0$, and $\phi(z, x') \neq 0$; one of them may be in direction of x . At any rate $\phi(\phi(z, x'), x)$ must not vanish unless $\phi(z, x')$ is in the direction of x . But by [3.6],

$$\begin{aligned}\phi(\phi(z, x'), x) &= \theta(\theta(x', x), z) - \theta(\theta(x, z), x') \\ &= \lambda_x \theta(x', z) - \lambda_x \theta(z, x') = 0.\end{aligned}$$

On the other hand, if $\phi(z, x')$ is in the direction of x , by taking any vector y in different direction we should have $\phi(\phi(z, x'), y) \neq 0$. Yet still [3.6] gives

$$\phi(\phi(z, x'), y) = \lambda_y \theta(x', z) - \lambda_y \theta(z, x') = 0,$$

a contradiction. Therefore the theorem.

A simplest example of an associative algebra in a diffused space is the case where the dimension of the space is 3. In it, there is a vector v which is not commutative with any vector in different direction, and there is a plane linearly independent of

v , in which all vectors are commutative with each other. By Theorem 20, the general form of $\theta(x, y)$ for any x and y is

$$[9.5] \quad \theta(x, y) = \theta \lambda_{yx} + \lambda_{xy}.$$

From [3.5] we have

$$[9.6] \quad \lambda \phi_{(x, y)} = 0 \text{ for any } x \text{ and } y.$$

By Theorem 7 and the Jacobian Identity [J] of §4, we can easily derive

$$[9.7] \quad \phi(x, y) = [u_1 x y] u_1 + [u_2 x y] u_2$$

for any x and y where u_1 and u_2 are two linearly independent vectors in that commutative plane.

Then by [3.6] we get

$$\begin{aligned} [9.8] \quad \lambda_x \lambda_{yz} - \lambda_z \lambda_{yx} &= [u_2 z x] [u_1 u_2 y] u_1 + [u_1 z x] [u_2 u_1 y] u_2 \\ &= [u_1 u_2 y] ([u_1 z x] u_2 + [u_2 u_1 x] z + [u_2 z u_1] x) \\ &\quad + [u_1 z x] [u_2 u_1 y] u_2 \\ &= [u_1 u_2 y] ([u_2 u_1 x] z + [u_2 z u_1] x) \end{aligned}$$

$$\therefore \lambda_x \lambda_y = -[u_1 u_2 y] [u_1 u_2 x], \text{ or } \lambda_x = \pm i [u_1 u_2 x] \text{ for all } x$$

So [9.5] becomes

$$[9.9] \quad \theta(x, y) = \pm i [u_1 u_2 y] x \pm i [u_1 u_2 x] y, \quad i = \sqrt{-1}.$$

Hence

$$\begin{aligned} [9.10] \quad \sigma(x, y) &= \pm i [u_1 u_2 y] x \pm i [u_1 u_2 x] y \\ &\quad + [u_1 x y] u_1 + [u_2 x y] u_2 \end{aligned}$$

or if we write $x = x^1 u_1 + x^2 u_2 + x^3 v$ and

$y = y^1 u_1 + y^2 u_2 + y^3 v$, we have

$$\begin{aligned} [9.11] \quad \sigma(x, y) &= [u_1 u_2 v] (\pm i y^3 x^1 \pm i x^3 y^1 + x^2 y^3 - x^3 y^2) u_1 \\ &\quad + [u_1 u_2 v] (\pm i y^3 x^2 \pm i x^3 y^2 + x^3 y^1 - x^1 y^3) u_2 \\ &= 2 [u_1 u_2 v] i x^3 y^3 v \\ &= K [(\pm i y^3 x^1 \pm i x^3 y^1 + x^2 y^3 - x^3 y^2) u_1 \\ &\quad + (\pm i y^3 x^2 \pm i x^3 y^2 + x^3 y^1 - x^1 y^3) u_2 \pm 2 i x^3 y^3 v] \end{aligned}$$

We see that σ vanishes on the plane containing u_1 and u_2 .

This algebra can be called *1-way non-commutative*. It can be generalized to n -dimensional with a complete abelian subspace of $n-1$ dimensions. From Theorem 21, it follows that there are no 2 or more way non-commutative associative algebras in spaces of dimension $n > 2$.

§ 10. RESUMÉ

Associative algebras in non-commutative space, as we have investigated so far, can be divided into three classes, namely,

1. The entirely non-commutative algebras, in which only elements of same direction are commutative with each other, or the skew part of the associative product $\sigma(x, y)$, i.e. $\phi(x, y)$, vanishes only when x and y are collinear. There is only one algebra essentially of this kind, that is, the one in 2-dimensional space. The symmetric part of $\sigma(x, y)$ is:

$$\theta(x, y) = \pm [ay]_x \pm [ax]_y \dots\dots\dots [4.3], \text{ §4.}$$

$$\text{and } \phi(x, y) = [x, y] a \dots\dots\dots [4.2], \text{ §4.}$$

$$\text{So } \sigma(x, y) = \pm [ay]_x \pm [ax]_y + [xy]a$$

$$= 2[ay]_x \text{ or } 2[ax]_y \dots\dots\dots [4.4], \text{ §4.}$$

2. The centralized algebras, in which $\phi(x, y)$ vanishes if and only if x and y are coplanar with any vector in the centrum. The possible spaces containing such algebras are of dimension n where $r+2 \leq n \leq 2r+2$; r being the dimension of the centrum. The general form of $\theta(x, y)$ is:

$$\theta(x, y) = \lambda_y x + \lambda_x y + K_{x,y} \dots\dots\dots [5.1], \text{ §5.}$$

Special cases of spaces of dimension 2, 3, and 4 are considered in this work. The results are:

1-way commutative:

$$\text{2-dimensional, } \phi(x, y) = 0$$

$$1. \sigma(x, y) = \theta(x, y) = -x^1 y^1 u - (x^1 y^2 + x^2 y^1) v \dots\dots [6.5], \text{ §6.}$$

$$2. \sigma(x, y) = \theta(x, y) = -(x^1 y^1 - x^2 y^2) u - (x^1 y^2 + x^2 y^1) v.$$

-[6.6], §6.
 (the algebra of complex numbers)
3. $\sigma(x, y) = \theta(x, y) = s(x, y)u = x^1y^1u$ [6.7], §6.
4. $\sigma(x, y) = \theta(x, y) = (x^1y^1 + x^2y^2 \pm ix^1y^2 \pm ix^2y^1)u \pm 2ix^2y^2v$.
 or $= (x^1y^1 - x^2y^2 \pm x^1y^2 \pm x^2y^1)u \pm 2x^2y^2v$
 or $= (ix^1y^1 + ix^2y^2 \pm x^1y^2 \pm x^2y^1)u \pm 2x^2y^2v$ [6.8], §6.
- 3-dimensional, $\phi(x, y) = [uxy]a$ [6.9], §6.
1. $\theta(x, y) = 0$, $\sigma(x, y) = \phi(x, y) = [uxy]u$ [6.12], §6.
2. $\theta(x, y) = x^2y^2s(v, v)u = x^2y^2u$
 $\sigma(x, y) = u((x^2y^2 + x^2y^3 - x^3y^2)[uvw])$
 $= (x^2y^2 + x^2y^3 - x^3y^2)u$ [6.13], §6.
3. $\theta(x, y) = (x^2y^2 + x^3y^3)u$
 $\sigma(x, y) = [x^2y^2 + x^3y^3 + (x^2y^3 - x^3y^2)[uvw]]u$ [6.14], §6.
4. $\theta(x, y) = \pm[ua y]x \pm [ua x]y + s(x, y)u$
 $= (\pm(u^3y^2 - u^2y^3)x^1 \pm (u^3x^2 - u^2x^3)y^1)a$
 $+ (\pm(u^3y^2 - u^2y^3)x^2 \pm (u^3x^2 - u^2x^3)y^2$
 $+ (x^2y^2 - x^3y^3)u^2)b$
 $+ (\pm(u^3y^2 - u^2y^3)x^3 \pm (u^3x^2 - u^2x^3)y^3$
 $+ (x^2y^2 - x^3y^3)u^3)c$
 $\sigma(x, y) = (\pm(u^3y^2 - u^2y^3)x^1 \pm (u^3x^2 - u^2x^3)y^1$
 $\pm (x^3y^1 - x^1y^3)u^2 \pm (x^1y^2 - x^2y^1)u^3)a$
 $+ (\pm(u^3y^2 - u^2y^3)x^2 \pm (u^3x^2 - u^2x^3)y^2$
 $+ (x^2y^2 - x^3y^3)u^2)b$
 $+ (\pm(u^3y^2 - u^2y^3)x^3 \pm (u^3x^2 - u^2x^3)y^3$
 $+ (x^2y^2 - x^3y^3)u^3)c$ [6.15], §6.
- $\phi(x, y) = \pm\{u^2(x^3y^1 - x^1y^3) + u^3(x^1y^2 - x^2y^1)\}a$
5. $\theta(x, y) = \pm y^2x \pm x^2y + (x^1y^1 - x^2y^2)u$
 $\sigma(x, y) = (\pm y^2x^1 \pm x^2y^1 + x^1y^1 - x^2y^2)u$
 $\pm 2x^2y^2v \pm 2x^2y^3a$ [6.16], §6.
6. $\theta(x, y) = -s(u, y)x - s(u, x)y + s(x, y)u$
 $= -(x^1y^1 + x^2y^2)u - (y^1x^2 + x^1y^2)v - (y^1x^3 + x^1y^3)a$
 $\sigma(x, y) = -(x^1y^1 + x^2y^2)u - (y^1x^2 + y^2x^1)v$

$$-(x^3y^1+x^1y^3\pm x^2y^3\mp x^3y^2)a\ldots\ldots\ldots[6.17], \S 6.$$

4-dimensional,

$$\begin{aligned} 1. \phi(x,y) &= [B_1 u x y] B_1 + [B_2 u x y] B_2 + [B_3 u x y] B_3 \\ &= (x^3y^2 - x^2y^3) B_1 + (x^1y^3 - x^3y^1) B_2 \\ &\quad + (x^2y^1 - x^1y^2) B_3 \end{aligned}$$

$$\begin{aligned} \theta(x,y) &= s(x,y)u - s(y,u)x - s(x,u)y \\ &= (x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4)u - (x^1y^4 + x^4y^1)B_1 \\ &\quad - (x^2y^4 + x^4y^2)B_2 - (x^3y^4 + x^4y^3)B_3 \end{aligned}$$

$$\begin{aligned} \sigma(x,y) &= (x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4)u \\ &\quad - (x^1y^4 + x^4y^1 + x^2y^3 - x^3y^2)B_1 \\ &\quad - (x^2y^4 + x^4y^2 + x^3y^1 - x^1y^3)B_2 \\ &\quad - (x^3y^4 + x^4y^3 + x^1y^2 - x^2y^1)B_3 \ldots \ldots \ldots [7.8], \S 7. \end{aligned}$$

$$\begin{aligned} 2. \phi(x,y) &= [B_1 u x y] B_1 + [B_2 u x y] B_2 + [B_3 u x y] B_3 \\ &= [x y u] \ldots \ldots \ldots [7.11], \S 7. \end{aligned}$$

$$\begin{aligned} \theta(x,y) &= \pm i[u B_1 B_2 y]x \pm i[u B_1 B_2 x]y + s(x,y)u \\ &\ldots \ldots \ldots [7.15], \S 7. \end{aligned}$$

$$\begin{aligned} \sigma(x,y) &= \pm i[u B_1 B_2 y]x \pm i[u B_1 B_2 x]y + s(x,y)u \\ &\quad + [x y u] \ldots \ldots \ldots [7.16], \S 7. \end{aligned}$$

$$s(x,y) = x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4 \ldots \ldots \ldots [7.17], \S 7.$$

2-way commutative: general results,

$$\theta(x,y) = \lambda_{yx} + \lambda_{xy} + s_1(x,y)u_1 + s_2(x,y)u_2 \ldots \ldots [8.1], \S 8.$$

3-dimensional, $\phi(x,y) = 0$

4-dimensional, $\phi(x,y) = [u_1 u_2 x y]a$

3. The diffused algebras which are not entirely non-commutative, nor centralized, in which there may be abelian subspaces not all of which are commutative with same set of elements. The general form of $\theta(x,y)$ is:

$$\theta(x,y) = \lambda_{yx} + \lambda_{xy} - K'' r_{xx} \ldots \ldots \ldots [9.3], \S 9.$$

A special case of the 3-dimensional space is discussed, which is 1-way non-commutative; the results are:

$$\theta(x,y) = \pm i[u_1 u_2 y]x \pm i[u_1 u_2 x]y \ldots \ldots \ldots [9.9], \S 9.$$

$$\phi(x,y) = [u_1 x y]u_1 + [u_2 x y]u_2 \ldots \ldots \ldots [9.7], \S 9.$$

$$\begin{aligned}\sigma(x,y) = & [u_1 u_2 v] [(\pm i y^3 x^1 \pm i x^3 y^1 + x^2 y^3 - x^3 y^2) u_1 \\ & + (\pm i y^3 x^2 \pm i x^3 y^2 + x^3 y^1 - x^1 y^3) u_2 \\ & \pm 2 i x^3 y^3 v] \dots\dots\dots [9.11], \S 9.\end{aligned}$$

In each class is given a Theorem concerning the general space. They are Theorem 10, §4, Theorem 12, §5, and Theorem 21, §9, which give, except for the third class, a clear view of all possible solutions. We also see from §3 that the Jacobian Identity [J] is necessary for an associative σ -operation; but in several places it shows the insufficiency, for instance, in 1-way commutative algebras of 4-dimensional space [J] may be fulfilled by a ϕ of the form [7.4] or [7.7], but without the restriction given by [6.3] that $\lambda_u \neq 0$, we would not have a possible associative algebra on account of the arising of a contradiction from [6.3] if $\lambda_u = 0$, as we have shown in §7, moreover, theorems 10, 12, and 21 definitely preclude the existence of certain associative algebras in which still there may exist ϕ -functions satisfying [J], through the panel of [3.6]

There are still problems open for investigation, such as the actual solutions for more general non-commutative associative algebras in centralized spaces of dimension n where $r+2 \leq n \leq 2r+2$, with a general r , the relation between the symmetric part and the skew part of the associative product in general, and more specified possible and actual solutions of diffused associative algebras.

The ultimate object of this work is to get hold of all existing non-commutative associative algebras to be classified by their different degree of non-commutativity, or by the different characteristics of the imbedding skew parts.

